Continuity and Differentiability

Hilary term 2007, Qian's Notes (Revised on 15 Feb., 2007)

Contents

[Monday 15 January 2007]
Numbering system: I don't use numbering system in my lectures. But I may quote a statement via the numbering system in the lecture notes. Thus, Theorem 1.43 refers to Theorem 1.43 on page 14, Lecture Notes.

Some notations.

- $\mathbb{C}$: set of all complex numbers – the complex plane;
- $\mathbb{R}$: set of real numbers – the real line; $\mathbb{R} \subset \mathbb{C}$.
- $\mathbb{Q}$: rational numbers, $\mathbb{Q} \subset \mathbb{R}$.
- $\forall$: "for all", "for every one", "whenever"
- $\exists$: "there exist(s)", "there is (are)",
- iff stands for "if and only if"

If $z = x + iy$ is a complex number, then its $|z| = \sqrt{x^2 + y^2}$ is called the *absolute value* of z (also called the modulus of z).

Chapter 1

Function Limits and Continuity

1.1 Function Limits

Sequence limits and completeness Recall the definition of limits for sequences from Michaelmas term.

Definition 1.1 *1) A sequence* $\{z_n\}$ *of real (or complex) numbers has limit* l*, denoted by* $z_n \to l$ *or* $\lim_{n\to\infty} z_n = l$*, if* $\forall\ \varepsilon > 0$*,* $\exists$ *a positive [integer] number* N *s. t.* $|z_n - l| < \varepsilon$ *(*$\forall$ $n > N$*).*
2) A sequence $\{z_n\}$ *of (real or complex) numbers converges if it has a limit* l*.*
3) $\{z_n\}$ *is called a* Cauchy sequence *if* $\forall\ \varepsilon > 0\ \exists$ *a positive number [integer]* N *s. t.*

$$|z_n - z_m| < \varepsilon \quad \text{whenever } n, m > N.$$

Remark 1.2 $\forall$ *means "for every"; "whenever"; "for all".* $\exists$ *means "there exist(s)"; "there is (are)".*
s. t. is the abbreviation of "such that", "iff" stands for "if and only if" and "resp." for "respectively".

Remark 1.3 *According to Definition 1), a sequence* $\{z_n\}$ does not *converge to* l *[i.e. either* $\{z_n\}$ *diverges or* $z_n \to a \neq l$*], iff* $\exists\ \varepsilon > 0$*, s.t.* $\forall$ *natural number* k*,* $\exists$ *[at least one]* $n_k > k$ *s. t.*

$$|z_{n_k} - l| \geq \varepsilon\ .$$

To formulate a contrapositive proposition: Replace $\forall$ by $\exists$, and $\exists$ by $\forall$, and negate the statement.

Theorem 1.4 *(**Cauchy's Criterion**, **The General Principle for Convergence**) A sequence* $\{z_n\}$ *of real (or complex) numbers converges iff it is a Cauchy sequence.*

In this sense, we say that the real line $\mathbb{R}$ [the blackboard bold letter $\mathbb{R}$ denotes the set of all real numbers] and the complex plane $\mathbb{C}$ [the set of all complex numbers] are *complete metric spaces.*

Remark 1.5 *According to Cauchy's criterion, $\{z_n\}$ diverges [i.e. has no finite limit], iff $\exists\ \varepsilon > 0$, s.t. $\forall$ natural number k, $\exists$ [at least] two integers $n_{k_1}, n_{k_2} > k$ s. t.*

$$|z_{n_{k_1}} - z_{n_{k_2}}| \geq \varepsilon\ .$$

Compactness The following theorem demonstrates the "compactness" of a bounded subset.

Theorem 1.6 ***(Bolzano-Weierstrass' Theorem)*** *Any bounded sequence in $\mathbb{R}$ (or in $\mathbb{C}$) has a subsequence which converges to a point in $\mathbb{R}$ (in $\mathbb{C}$).*

The proofs of theorems about continuous functions we are going to prove in this course rely on the following

Corollary 1.7 *A bounded sequence $\{z_n\}$ in $\mathbb{R}$ (or in $\mathbb{C}$) converges to a limit l iff all convergent subsequences of $\{z_n\}$ have the same limit l.*

Proof. ($\Longrightarrow$ "only if " part: Necessity) Proved in Analysis I: any subsequence of a convergent sequence tends to the same limit.
($\Longleftarrow$ "if" part: Sufficiency) Argue by contradiction [*If you can not prove a statement directly, then formulate the contrapositive, and prove it is wrong*]. Suppose $\{z_n\}$ is divergent. Since $\{z_n\}$ is bounded, $\exists$ a subsequence $\{z_{n_k}\}$ converging to some limit l_1 [Bolzano-Weierstrass' Theorem]. Let $\{y_n\} \equiv \{z_n\} \setminus \{z_{n_k}\}$. If $\{y_n\}$ didn't converge to l_1 [otherwise $z_n \to l_1$: Exercise]. Therefore, $\exists\ \varepsilon_0 > 0$ s.t. $\forall$ integer j $\exists$ a natural number $n_j > j$

$$|y_{n_j} - l_1| \geq \varepsilon_0\ .$$

[*The contrapositive to that $y_n \to l_1$*]. Since $\{y_{n_j}\}$ is bounded, by Bolzano-Weierstrass' Theorem**,** $\exists$ a convergent subsequence $\{z'_{n_k}\}$ of $\{y_{n_j}\}$: $\lim z'_{n_k} = l_2$. Since

$$|z'_{n_k} - l_1| \geq \varepsilon_0 \qquad \forall k$$

so that

$$\lim_{k\to\infty} |z'_{n_k} - l_1| = |l_2 - l_1| \geq \varepsilon_0 > 0\ .$$

Therefore $l_1 \neq l_2$, and we thus have found two subsequences of $\{z_n\}$ which converge to distinct limits. ■

Limit points

Definition 1.8 *Let $E \subseteq \mathbb{R}$ (resp. $\mathbb{C}$). $p \in \mathbb{R}$ (resp. $\mathbb{C}$) is called a* limit point *(or an* accumulation point*, a* cluster point *) of E, if $\forall\ \varepsilon > 0$, there is at least one point $z \in E$ other then p such that*

$$(0 <\ )|z - p| < \varepsilon$$

A point which is not a limit point of E is called an isolated point *of E.*

Proposition 1.9 *$p \in \mathbb{R}$ is a limit point of an interval $[a, b]$ ($(a, b]$, $[a, b)$ or (a, b)) iff $p \in [a, b]$.*

[Exercise]

Functions

A real (resp. complex)-valued function f on $E \subset \mathbb{R}$ (or $\mathbb{C}$) is a correspondence which assigns each x in E to a unique real (resp. complex) number $f(x)$. E is called the domain of f.

Example 1.10 *$f(x) = \sqrt{1-x^2}$ with domain $E = [-1, 1]$. What is its graph? Its graph looks continuous.*

Example 1.11 *Consider function f on $E = (0, 1]$*

$$f(x) = \begin{cases} \frac{p}{q}, & \text{if } x = \frac{p}{q} \text{ and } (p, q) = 1, \\ 0, & \text{if } x \text{ is irrational} \end{cases}.$$

Can't sketch its graph. [Try with Maple].

Example 1.12 *$f(x) = x \sin \frac{1}{x}$ with domain $\mathbb{R}\backslash\{0\}$. As x tends to 0, f oscillates but goes to 0. f has limit 0 as x goes to 0.*

Definition 1.13 *Let $E \subseteq \mathbb{R}$ (or $\mathbb{C}$), and $f : E \to \mathbb{R}$ (or $\mathbb{C}$) a real (or complex) function. Let p be a* limit point *of E [p is not necessary in E] and l be a number. If $\forall\ \varepsilon > 0\ \exists$ a number $\delta > 0$ [which may depend on p and ε] s. t.*

$$|f(x) - l| < \varepsilon \quad \text{whenever } x \in E \text{ such that } 0 < |x - p| < \delta$$

then we say f tends to l as x goes to p [in E], written as

$$\lim_{x \to p} f(x) = l$$

or $f(x) \to l$ as $x \to p$ [in E].

[Wednesday 17 January, 2007]

Recall the definition of function limits.

Definition. Let $f : E \to \mathbb{R}$ (or $\mathbb{C}$) where E is a subset of $\mathbb{R}$ (or $\mathbb{C}$), and p is a limit point of E. Then $f(x) \to l$ as $x \to p$, if $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$$|f(x) - l| < \varepsilon \qquad \forall x \in E \text{ such that } 0 < |x - p| < \delta\ .$$

[Do a sketch to demonstrate the meaning].

Example 1.14 *Let $f(x) = |x|^\alpha \sin \frac{1}{x}$ for $x \neq 0$ where $\alpha > 0$ is a constant. [$E = \mathbb{R}\backslash\{0\}$]. Show that $f(x) \to 0$ as $x \to 0$.*

Since $\left|x^\alpha \sin \frac{1}{x}\right| \le |x|^\alpha$ for any $x \neq 0$. Therefore, $\forall\ \varepsilon > 0$, choose $\delta = \varepsilon^{1/\alpha}$. Then

$$\left|x^\alpha \sin \frac{1}{x} - 0\right| \le |x|^\alpha < \varepsilon$$

whenever $0 < |x - 0| < \delta$. According to definition, $|x|^\alpha \sin \frac{1}{x} \to 0$ as $x \to 0$.

Remark 1.15 f doesn't converge *to* l *as* $x \to p$ *[i.e. either* f *has no limit or* $f(x) \to a \neq l$ *as* $x \to p$*], then* $\exists\, \varepsilon > 0$*, s.t.* $\forall\, \delta > 0$*,* $\exists\, x \in E$ *such that* $0 < |x-p| < \delta$ *but* $|f(x) - l| \geq \varepsilon$*. [Not show in the lecture].*

Proposition 1.16 *Let* $f : E \to \mathbb{R}$ *(or* $\mathbb{C}$*) and* p *be a limit point of* E*. If* f *has a limit as* $x \to p$*, then the limit is unique.*

Suppose $f(x) \to l_1$ and also $f(x) \to l_2$ as $x \to p$, where $l_1 \neq l_2$, then $\frac{1}{2}|l_1 - l_2| > 0$. By definition, there is $\delta_1 > 0$ such that

$$|f(x) - l_1| < \frac{1}{2}|l_1 - l_2| \text{ whenever } x \in E \text{ such that } 0 < |x-p| < \delta_1 .$$

Similarly, $\exists \delta_2 > 0$ s.t.

$$|f(x) - l_2| < \frac{1}{2}|l_1 - l_2| \quad \text{whenever } x \in E \text{ such that } 0 < |x-p| < \delta_2 .$$

Let $\delta = \delta_1 \wedge \delta_2$. Since p is a limit point of E, there is at least $x_0 \in E$ such that $0 < |x_0 - p| < \delta$. However

$$\begin{aligned} |l_1 - l_2| &= |f(x) - l_1 - (f(x) - l_2)| && \text{[+1 and -1 technique]} \\ &\leq |f(x) - l_1| + |f(x) - l_2| && \text{[Triangle Ineq.]} \\ &< \frac{1}{2}|l_1 - l_2| + \frac{1}{2}|l_1 - l_2| \\ &= |l_1 - l_2| \end{aligned}$$

a contradiction.

Theorem 1.17 *Let* $f : E \to \mathbb{R}$ *(or* $\mathbb{C}$*) where* $E \subseteq \mathbb{R}$ *(or* $\mathbb{C}$*),* p *be a limit point of* E *and* $l \in \mathbb{C}$*. Then* $\lim_{x \to p} f(x) = l$ *iff* $\forall$ *sequence* $\{p_n\}$ *in* E *s. t.* $p_n \neq p$ *and* $\lim_{n\to\infty} p_n \to p$ *we have*

$$\lim_{n \to \infty} f(p_n) = l .$$

[$\lim_{x \to p} f(x) = l$ iff f tends to the same limit l along any sequence in E going to p.]
$\Longrightarrow$ **"only if part.** Suppose $\lim_{x\to p} f(x) = l$. Then $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s. t.

$$|f(x) - l| < \varepsilon \qquad \forall x \in E \text{ s.t. } 0 < |x-p| < \delta .$$

Now suppose $\{p_n\}$ is a sequence in E, $p_n \to p$ and $p_n \neq p$. Then $\exists$ a natural number N s. t.

$$|p_n - p| < \delta \quad \forall\, n > N .$$

Since $p_n \neq p$, so that

$$|f(p_n) - l| < \varepsilon \qquad \forall\, n > N.$$

Hence, $\lim_{n\to\infty} f(p_n) = l$.
[$\Longleftarrow$ "if" part] Argue by contradiction. Suppose $\lim_{x\to p} f(x) = l$ is not true, then $\exists \varepsilon > 0$, s. t. $\forall \delta = 1/n$, $\exists$ at least one point $x_n \in E$, $0 < |x_n - p| < 1/n$ but

$$|f(x_n) - l| \geq \varepsilon_0 .$$

Therefore we have found a sequence $\{x_n\}$ which converges to p but $\{f\,(x_n)\}$ does not tend to l. Contradiction. ■

Proposition 1.18 *[Algebra of limits] Let p be a limit point of E, and f, g be two real (or complex) functions on E. Suppose* $\lim_{x\to p} f(x) = A$ *and* $\lim_{x\to p} g(x) = B$. *Then*

Corollary 1.19 *1)* $\lim_{x\to p}(f(x) \pm g(x)) = A \pm B$*;*
2) $\lim_{x\to p}(f(x)g(x)) = AB$ *;*
3)if $B \neq 0$,

$$\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{A}{B} \ .$$

[Follows the previous theorem and the algebra of limits for sequences. Question in the problem sheet].

Example 1.20 *Show that* $\lim_{x\to 0} \sin\frac{1}{x}$ *doesn't exist.*

Indeed, let $x_n = \frac{1}{\pi n}$ and $y_n = \frac{1}{\pi n + \pi/2}$. Then both sequences x_n and y_n tend to 0, but

$$\lim_{n\to\infty} \sin\frac{1}{x_n} = 0$$

and

$$\lim_{n\to\infty} \sin\frac{1}{y_n} = 1 \ .$$

So $\lim_{x\to 0} \sin\frac{1}{x}$ cannot exist [Theorem 1.17].

Example 1.21 *If* $\lim_{x\to p} f(x) = l \neq 0$*, then there is a positive number* $\delta > 0$ *such that*

$$|f(x)| \geq \frac{|l|}{2} \qquad \forall x \in E \ s.t. \ \ 0 < |x-p| < \delta \ .$$

In particular, $|f(x)| > 0$ *for all* $x \in E$ *s.t.* $0 < |x-p| < \delta$.

Since $\lim_{x\to p} f(x) = l$ and $|l| > 0$, there is a number $\delta > 0$ such that

$$|f(x) - l| < \frac{|l|}{2} \quad \text{whenever } x \in E,\ \ 0 < |x-p| < \delta \ .$$

By the Triangle Ineq.

$$\begin{aligned} |f(x)| &= |l - (f(x) - l)| \\ &\geq |l| - |f(x) - l| \\ &> |l| - \frac{|l|}{2} = \frac{|l|}{2} \end{aligned}$$

for any $x \in E$ such that $0 < |x-p| < \delta$.
[Monday 22 January, 2007]

1.2 Continuity of functions

In the definition of $\lim_{x\to p} f(x)$, the point p may not belong to the domain E of f. Even $f(p)$ is well-defined, the limit of f at p may have nothing to do with $f(p)$.

Definition 1.22 *Let $f : E \to \mathbb{R}$ (or $\mathbb{C}$), where $E \subseteq \mathbb{R}$ (or $\mathbb{C}$), and $p \in E$ [p belongs to the domain of f]. If $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ s. t.*

$$|f(x) - f(p)| < \varepsilon \quad \forall x \in E \text{ s.t. } |x - p| < \delta,$$

then we say that f is continuous at p.

f is continuous at any isolated point of E.
If p is a limit point of E, then f is continuous at p, if and only if

1. p belongs to the domain of f, i.e. $f(p)$ is well-defined;
2. $\lim_{x\to p} f(x)$ exists;
3. and $\lim_{x\to p} f(x)$ equals the value of f at p.

Example 1.23 *Let $\alpha > 0$ be a constant. The function $f(x) = |x|^\alpha \sin\frac{1}{x}$ not continuous at $x = 0$ as f is not well-defined. Let*

$$g(x) = \begin{cases} |x|^\alpha \sin\frac{1}{x}, & \text{if } x \neq 0\ ; \\ 0, & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$.

Example 1.24 *Let $f : (0, 1] \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} \frac{p}{q}\ , & \text{if } x = \frac{p}{q} \text{ and } (p, q) = 1, \\ 0\ , & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is continuous at irrationals of $(0, 1]$, not continuous at rationales. [Question 1, Problem Sheet 2].

Suppose $x_0 \in (0, 1)$ is an irrational number, so that $f(x_0) = 0$. Then for any $\varepsilon > 0$,

$$|f(x) - f(x_0)| \leq \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \frac{1}{q}\ ; & \text{if } x = \frac{p}{q} \text{ and } (p, q) = 1\ . \end{cases}$$

While, there are only finite many pair of positive integers p and q such that $p \leq q$ and $q \leq \frac{1}{\varepsilon}$. Let

$$\delta = \min\left\{\left|x_0 - \frac{p}{q}\right| : p \leq q \text{ and } q \leq \frac{1}{\varepsilon}\right\} > 0$$

Then,

$$|f(x) - f(x_0)| < \varepsilon$$

for any x such that $|x - x_0| < \delta$. Thus f is continuous at x_0.
If $x_0 = \frac{p}{q} \in (0, 1]$ is a rational number, then, for $\varepsilon = \frac{p}{2q} > 0$ and for whatever how small $\delta > 0$, there is a irrational number $x \in (0, 1]$ such that $|x - \frac{p}{q}| < \delta$ and

$$|f(x) - f(x_0)| = \frac{p}{q} > \varepsilon \ .$$

f is not continuous at rational numbers.

Proposition 1.25 *If f and g are continuous at p, then so are $f \pm g$; fg and f/g (provided $g(p) \neq 0$).*

[Definition + Algebra of function limits].

Theorem 1.26 *If $f : E \to \mathbb{C}$ and $g : f(E) \to \mathbb{C}$, then define $h : E \to \mathbb{C}$ by*

$$h(x) = (g \circ f)(x) \equiv g(f(x)) \qquad \textit{for } x \in E.$$

If f is continuous at $p \in E$ and g is continuous at $f(p)$, then h is continuous at p.

[Composition of two continuous functions is continuous.] [Definition + Question 2, Problem Sheet 1].
Proof. For any $\varepsilon > 0$, since g is continuous at $f(p)$, there is a number $\delta_1 > 0$ such that

$$|g(y) - g(f(p))| < \varepsilon \quad \forall y \in f(E) \text{ s.t. } |y - f(p)| < \delta_1.$$

That is

$$|g(f(x)) - g(f(p))| < \varepsilon \ \ \forall x \in E \text{ s.t. } |f(x) - f(p)| < \delta_1.$$

However, f is continuous at p, $\exists\ \delta > 0$ such that

$$|f(x) - f(p)| < \delta_1 \quad \forall x \in E \text{ s.t. } |x - p| < \delta \ .$$

Hence

$$|g(f(x)) - g(f(p))| < \varepsilon \quad \forall x \in E \text{ s.t. } |x - p| < \delta$$

so that h is continuous at p. ■

Example 1.27 *Let $f : \mathbb{C} \to \mathbb{C}$ (or $\mathbb{R} \to \mathbb{R}$) be a polynomial. Then f is continuous in $\mathbb{C}$ (or $\mathbb{R}$).*

1.3 Continuous functions on bounded intervals

In this part we are going to establish several important theorems about continuous functions on bounded intervals.

1.3.1 Uniform Continuity

Let's look at two examples.

Example 1.28 *Show that for every* $x_0 \neq 0$, $\lim_{x\to x_0} \frac{1}{x} = \frac{1}{x_0}$, *and thus* $\frac{1}{x}$ *is continuous at any* $x \neq x_0$.

Since

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x-x_0|}{|x||x_0|} \ ,$$

thus, if $|x - x_0| < \frac{|x_0|}{2}$ [so we should choose δ smaller then $\frac{|x_0|}{2}$], then

$$|x| \geq |x_0| - |x - x_0| > \frac{|x_0|}{2} \quad \text{[Triangle inequality]}$$

and therefore

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x-x_0|}{|x||x_0|} \leq \frac{2}{|x_0|^2}|x-x_0| \ .$$

[Thus in order to ensure $\left|\frac{1}{x} - \frac{1}{x_0}\right| < \varepsilon$ we only need $\frac{2}{|x_0|^2}|x-x_0| < \varepsilon$ and $|x-x_0| < \frac{|x_0|}{2}$.] Therefore, $\forall\ \varepsilon > 0$, choose $\delta = \min\left\{\frac{|x_0|}{2}, \frac{\varepsilon|x_0|^2}{2}\right\}$ [which is positive as $x_0 \neq 0$]. Then

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < \varepsilon$$

whenever $|x - x_0| < \delta$. Hence $\frac{1}{x} \to \frac{1}{x_0}$ as $x \to x_0$. Note that δ not only depends on ε but also on x_0.

Definition 1.29 *Let* $f : E \to \mathbb{R}$ *(or* $\mathbb{C}$*).* f *is uniformly continuous on* E*, if* $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ *s. t.*

$$|f(z) - f(x)| < \varepsilon \ , \qquad \forall\ z \in E \text{ s.t. } |z - x| < \delta$$

for all $x \in E$.

Example 1.30 *Suppose that* f *is Lipschitz continuous in* E*: there is a constant* M *such that*

$$|f(x) - f(y)| \leq M|x-y| \qquad \forall x, y \in E$$

Then f *is uniformly continuous at any point* $x \in E$*. For example,* $f(x) = \sqrt{x}$ *is Lipschitz continuous on* $[1, \infty)$*, so it is uniformly continuous.*

Let $x \in E$. For every $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M+1}$. Then

$$\begin{aligned} |f(x) - f(y)| &\leq M|x-y| \\ &\leq M\left(\frac{\varepsilon}{M+1}\right) < \varepsilon \end{aligned}$$

whenever $y \in E$ s.t. $|y - x| < \delta$. [δ doesn't depend on x. For a given $\varepsilon > 0$ we can find a δ that works for all x].

Theorem 1.31 *If $f : [a, b] \to \mathbb{R}$ (or $\mathbb{C}$) is continuous, then f is uniformly continuous.*

[A continuous function on a *closed interval* (a *compact set*) is uniformly continuous.]
Proof. Suppose that f were not uniformly continuous, then, $\exists\, \varepsilon > 0$, s. t. for any n [with $\delta = \frac{1}{n}$], $\exists$ a pair of points x_n, $y_n \in [a, b]$, $|x_n - y_n| < \frac{1}{n}$ but

$$|f(x_n) - f(y_n)| \geq \varepsilon \ .$$

[contrapositive to the uniform continuity]. Since $\{x_n\}$ is bounded, we can extract a subsequence $\{x_{n_k}\}$ from $\{x_n\}$ which converges to some p [Bolzano-Weierstrass' Theorem]. p must be a limit point of $[a, b]$, and therefore $p \in [a, b]$. Since

$$\begin{aligned} |y_{n_k} - p| &\leq |x_{n_k} - y_{n_k}| + |x_{n_k} - p| \\ &< \frac{1}{n_k} + |x_{n_k} - p| \to 0 \end{aligned}$$

Thus $x_{n_k} \to p$ and $y_{n_k} \to p$, so that

$$0 < \varepsilon \leq \lim_{k \to \infty} |f(x_{n_k}) - f(y_{n_k})| = |f(p) - f(p)| = 0$$

as f is continuous at p, which is impossible. ■

Remark 1.32 *The bounded closed interval $[a, b]$ can be replaced by any bounded and closed subset of $\mathbb{R}$ or $\mathbb{C}$, see W. Rudin's Principles, page 91, Theorem 4.19.*

Example 1.33 *$f(x) = \sqrt{x}$ is uniformly continuous in the unbounded interval $[0, +\infty)$.*

First show that $\sqrt{x}$ is continuous on $[0, 1]$ [Exercise], thus it must be uniformly continuous [Theorem 1.31]. On the other hand, $\sqrt{x}$ is Lipschitz continuous on $[1, \infty)$, so it is uniformly continuous on $[1, \infty)$.
Hence $\forall \varepsilon > 0$, $\exists \delta_1 > 0$ s. t.

$$|\sqrt{x} - \sqrt{y}| < \frac{\varepsilon}{2}, \qquad \forall x, y \in [0, 1] \text{ s.t. } |x - y| < \delta_1 \ . \tag{1.1}$$

and $\exists \delta_2 > 0$

$$|\sqrt{x} - \sqrt{y}| < \frac{\varepsilon}{2} \qquad \forall x, y \geq 1 \text{ s.t. } |x - y| < \delta_2 \ . \tag{1.2}$$

Choose $\delta = \min\{\delta_1, \delta_2, \frac{1}{2}\}$. Then, if $|x - y| < \delta$ but $x \in [0, 1]$ and $y \geq 1$. Then $|x - 1| < \delta$ and $|y - 1| < \delta$ so that

$$\begin{aligned} |\sqrt{x} - \sqrt{y}| &\leq |\sqrt{x} - \sqrt{1}| + |\sqrt{y} - \sqrt{1}| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence we have

$$|\sqrt{x} - \sqrt{y}| < \varepsilon$$

whenever x, $y \in [0, \infty)$ such that $|x - y| < \delta$. By definition, $f(x) = \sqrt{x}$ is uniformly continuous in the *unbounded interval* $[0, +\infty)$.

1.3.2 Boundedness

A real (or complex) function f in E is bounded if there is non-negative constant M such that

$$|f(z)| \leq M \qquad \forall\, z \in E\ .$$

In this case we also say f is bounded by M on E, or M is a *bound* of f on E.

Theorem 1.34 *If $f : [a,b] \to \mathbb{R}$ (or $\mathbb{C}$) is continuous, then f is bounded.*

Proof. By contradiction. Suppose f were unbounded, then for any natural number n, there is at least one point $x_n \in [a,b]$ such that $|f(x_n)| \geq n$. By Bolzano-Weierstrass' Theorem, if necessary by extracting a subsequence, we may assume that $x_n \to p \in [a,b]$. By assumption, f is continuous, so that

$$\lim_{n\to\infty} f(x_n) = f(p)\ ,$$

[Theorem 1.17], which implies that $\lim_{n\to\infty} |f(x_n)| = |f(p)|$, so that $\{f(x_n)\}$ is a bounded sequence [from Analysis I: convergent sequences are bounded], which is a contradiction to the fact that $|f(x_n)| \geq n$ for every n. Therefore f must be bounded. ■

If $f : E \to \mathbb{R}$ is a bounded, *real-valued* function on E, then we use

$$\sup_{x\in E} f(x) \quad \text{and} \quad \inf_{x\in E} f(x)$$

to denote the least upper bound and the greatest lower bound, and called the supremum and the infimum of f on E, respectively. [Not good for complex functions]. The existence of the least and the greatest bounds for a bounded real function f is guaranteed by the completeness of the real number system.

By definition, $M = \sup_{x\in E} f(x)$ if and only if $f(z) \leq M$ [so M is a *upper bound* on E] and $\forall\, \varepsilon > 0\ \exists\ z_\varepsilon \in E$ such that $f(z_\varepsilon) > M - \varepsilon$ [that is, any real which is smaller then M can not be a upper bound of f on E]. Similarly, $m = \inf_{x\in E} f(x)$ if and only if $f(z) \geq m$ (so m is a *lower bound* on E) and for every $\varepsilon > 0$ there is $z_\varepsilon \in E$ such that $f(z_\varepsilon) < m + \varepsilon$ [that is, any real which is greater then m can not be a lower bound of f on E].

Theorem 1.35 *If $f : [a,b] \to \mathbb{R}$ is continuous [not good for complex functions], then f achieves its supremum and infimum. That is, there are [at least] two points x_1 and x_2 in $[a,b]$ such that*

$$f(x_1) = \sup_{x\in[a,b]} f(x) \quad \textit{and} \quad f(x_2) = \inf_{x\in[a,b]} f(x)$$

respectively.

[sup and inf are attained. x_1, x_2 may be not unique. "A continuous function on a closed bounded interval is bounded and attains its bounds".]

Proof. **(1st Proof)** f is bounded [Theorem 1.34], so that $m \equiv \inf_{x\in[a,b]} f(x)$ exists by the completeness of the real number system [Analysis I]. Then $f(x) \geq m$ for all $x \in [a,b]$, and $\forall n$, $\exists\ x_n \in [a,b]$ such that

$$m \leq f(x_n) \leq m + \frac{1}{n}$$

[apply the definition of infimum to $\varepsilon = \frac{1}{n}$]. $\{x_n\}$ is bounded, we may thus extract a convergent subsequence $\{x_{n_k}\}$ [Bolzano-Weierstrass' Theorem]: $x_{n_k} \to p$. Then $p \in [a,b]$ [CLOSED interval]. Since f is continuous at p, $\lim_{x\to p} f(x) = f(p)$, so that thus $f(x_{n_k}) \to f(p)$ [Theorem 1.17]. While

$$m \leq f(x_{n_k}) \leq m + \frac{1}{n_k} \tag{1.3}$$

for all k. Letting $k \to \infty$ in equation (1.3) we obtain

$$m \leq \lim_{k\to\infty} f(x_{n_k}) = f(p) \leq \lim_{k\to\infty}\left(m + \frac{1}{n_k}\right) = m$$

so that $f(p) = m = \inf_{x\in[a,b]} f(x)$.
(2nd Proof) [By contradiction] Let us prove that the supremum of f is attained by contradiction. Suppose

$$f(z) < \sup_{x\in[a,b]} f(x) \qquad \forall z \in [a,b]\ .$$

Consider function g on $[a,b]$ defined by

$$g(x) = \frac{1}{M - f(x)}$$

which is positive and continuous on $[a,b]$. Therefore g is bounded (by Theorem 1.34) on $[a,b]$, say with bound M_0:

$$g(x) = \frac{1}{M - f(x)} \leq M_0.$$

It follows thus

$$f(x) \leq M - \frac{1}{M_0} < M$$

which is a contradiction. ■

Remark 1.36 *In the proofs of Theorem 3.21.31, 1.34 and 1.35, we have only used the following facts:*
1) $[a,b]$ is bounded;
2) $[a,b]$ is closed (i.e. $[a,b]$ contains all limit points of $[a,b]$;
3) f is continuous.

Definition 1.37 *A subset A of $\mathbb{R}$ (or $\mathbb{C}$) is bounded if there is a constant M, such that $|x| \leq M$ for all $x \in A$. A is closed if A contains all its limit points. A bounded and closed set in $\mathbb{R}$ (or in $\mathbb{C}$) is called a compact set in $\mathbb{R}$ (or C).*

With this definition and the previous remark we may reformulate what we have proven as the following

Theorem 1.38 *1) [Theorem 1.31 and 1.34] If f is a continuous real or complex function in a compact set E, then f is bounded and uniformly continuous in E.*
2) [Theorem 1.35] If f is a continuous real-valued function in a compact set E, then f attains its bounds, i.e. $\exists\ x_1,\ x_2 \in E$ such that

$$f(x_1) = \inf_{x\in E} f(x) \qquad \text{and} \qquad f(x_2) = \sup_{x\in E} f(x)\ .$$

Theorem 1.39 *(**The Intermediate Value Theorem (IVT)**). Let $f : [a, b] \to \mathbb{R}$ be continuous, and let c be a number between $f(a)$ and $f(b)$. Then there is at least one $\xi \in [a, b]$ such that $f(\xi) = c$.*

[One of the most important theorems in this course.]
Proof. By considering $-f$ instead of f if necessary, we may assume that $f(a) < c < f(b)$

[the case that $c = f(a)$ or $c = f(b)$ is trivial]. Let $g(x) = f(x) - c$. Then $g(a) < 0 < g(b)$. Let $x_1 = a$ and $y_1 = b$. Divide the interval $[x_1, y_1]$ into two equal parts. If $g(\frac{1}{2}(x_1+y_1)) = 0$ then $\xi = \frac{1}{2}(x_1 + y_1)$ will do. Otherwise, we choose $x_2 = x_1$ and $y_2 = (\frac{1}{2}(x_1 + y_1)$ if $g(\frac{1}{2}(x_1+y_1)) > 0$, or $x_2 = \frac{1}{2}(x_1+y_1)$ and $y_2 = y_1$ if $g(\frac{1}{2}(x_1+y_1)) < 0$. Then $g(x_2)g(y_2) < 0$; $[x_2, y_2] \subset [x_1, y_1]$ and

$$|y_2 - x_2| = \frac{1}{2}(b - a) \ .$$

Apply the previous argument to $[x_2, y_2]$ instead of $[a, b]$, we then find $[x_3, y_3] \subset [x_2, y_2]$;

$$|y_3 - x_3| = \frac{1}{2}|y_2 - x_2|$$

and $g(x_3)g(y_3) \le 0$. By repeating the same procedure, we thus find two sequences x_n, y_n, such that
1) either $g(x_n) = 0$ (or $g(y_n) = 0$), or $g(x_n)g(y_n) < 0$;
2) $[x_{n+1}, y_{n+1}] \subset [x_n, y_n]$ for any $n = 1, 2, \cdots$. That is $\{[x_n, y_n]\}$ is a net of closed intervals which becomes finer and finer;
3) since each time we break the previous interval $[x_n, y_n]$ into two equal parts to obtain $[x_{n+1}, y_{n+1}]$,

$$\begin{aligned} |y_n - x_n| &= \frac{1}{2}|y_{n-1} - x_{n-1}| \\ &= \cdots = \frac{1}{2^{n-1}}|y_1 - x_1| \\ &= \frac{b-a}{2^{n-1}} \ . \end{aligned}$$

Obviously, $\{x_n\}$ is a bounded increasing sequence, and $\{y_n\}$ is a bounded decreasing sequence, and thus $x_n \to \xi$ and $y_n \to \xi'$ for some ξ, $\xi' \in [a, b]$ [Analysis I: bounded monotone sequences converge]. Since

$$\lim_{n\to\infty} |y_n - x_n| = \lim_{n\to\infty} \frac{1}{2^{n-1}}(b - a) = 0,$$

so that $\xi = \xi'$. Since g is continuous at ξ, so that

$$0 \ge \lim_{n\to\infty} g(x_n)g(y_n) = \lim_{n\to\infty} g(x_n) \lim_{n\to\infty} g(y_n) = g(\xi)^2 \ .$$

Hence $g(\xi)^2 = 0$ so that $g(\xi) = 0$ [As $g(\xi)$ is a real number]. That is, $f(\xi) = c$. ■

Remark 1.40 *The above proof of the IVT also provides a method of finding roots to $f(\xi) = c$, but other methods may find roots faster if additional information about f (e.g. that f is differentiable) is available.*

Remark 1.41 *From the proof we can see that, if* $[x_n, y_n]$ *is a decreasing net of closed intervals (i.e.* $[x_n, y_n] \subset [x_{n+1}, y_{n+1}]$ *for each* n*) such that the length* $y_n - x_n \to 0$*, then* $\cap_{n=1}^{\infty}[x_n, y_n]$ *exactly contains one point (and in particular is not empty).*

Remark 1.42 *The proofs of Theorem 1.31, 1.34, 1.35 and IVT rely on the compactness of the closed interval* $[a, b]$ *[Bolzano-Weierstrass' theorem], and the proof of IVT requires more then that, We have indeed used the fact that* $[a, b]$ *is unbroken. That is, we have used the fact that* $[a, b]$ *is "connected". For details about"connectedness", see W. Rudin's Principles, page 93, Theorem 4.22 and Theorem 4.23.*

Corollary 1.43 *Let* $f : [a, b] \to \mathbb{R}$ *be continuous, and let* $M = \sup_{x\in[a,b]} f(x)$, $m = \inf_{x\in[a,b]} f(x)$*. Then for any* $c \in [m, M]$ *there is at least one* $\xi \in [a, b]$ *such that* $f(\xi) = c$*. Therefore*

$$f([a, b]) = [m, M] \ .$$

[A continuous real-valued function maps an interval into an interval.]
Proof. By definition, $m \le f(x) \le M$ for all $x \in [a, b]$, so that

$$f([a, b]) \equiv \{f(x) : x \in [a, b]\} \subseteq [m, M] \ .$$

On the other hand, f attains its bounds on $[a, b]$ [Theorem 1.35], there are (at least) two points $x_1, x_2 \in [a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$. For every $c \in [m, M]$, $\exists \xi \in [x_1, x_2] \subseteq [a, b]$ (or $[x_2, x_1]$ if $x_2 < x_1$]) such that $f(\xi) = c$ [apply IVT to f on the interval $[x_1, x_2]$], hence $[m, M] \subseteq f([a, b])$. ■

1.3.3 Monotonic Functions and Inverse Function Theorem

Theorem 1.44 *Let* f *be a continuous real function on* $[a, b]$*. Suppose further that* f *is one-to-one on* $[a, b]$ *so that the inverse function* f^{-1} *exists. Then* f^{-1} *is continuous on* $f([a, b])$*.*

Proof. By Corollary 1.43, $f([a, b]) = [m, M]$, where m and M are the minimal and the maximal values of f respectively, and thus $f([a, b])$ is also a bounded, closed interval. Let $y \in f([a, b])$, and take any sequence $\{y_n\}$ in $f([a, b])$ such that $y_n \to y$. If we can show that $f^{-1}(y_n) \to f^{-1}(y)$, then $\lim_{z\to y} f^{-1}(z) = f^{-1}(y)$ [Theorem 1.17] and thus f^{-1} is continuous at y. Let $x = f^{-1}(y)$ and $x_n = f^{-1}(y_n)$. Since $\{x_n\}$ is bounded, to prove that $x_n = f^{-1}(y_n) \to x = f^{-1}(y)$, we only need to show that any convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has the same limit x [Corollary 1.7]. Let $\lim_{k\to\infty} x_{n_k} = \hat{x}$. Then, since f is continuous at $\hat{x}$, $f(x_{n_k}) \to f(\hat{x})$ [Theorem 1.17]. On the other hand, $f(x_{n_k}) = y_{n_k} \to y = f(x)$ [Corollary 1.7], so that $f(\hat{x}) = f(x)$. Since f is one-to-one, $x = \hat{x}$, and therefore $\{x_{n_k}\}$ has [the same] limit x. ■

Remark 1.45 *The conclusion of the above theorem is still valid if we replace* $[a, b]$ *by any compact subset of the complex plane* $\mathbb{C}$*, see T. M. Apostol: Mathematical Analysis (2nd Edition), Theorem 4.29 (page 83).*

Definition 1.46 *Let f be a real function on $E \subseteq \mathbb{R}$.*
1) If $f(x) \leq f(y)$ (resp. $f(x) \geq f(y)$) whenever $x < y$ and $x, y \in E$, then we say f is increasing (resp. decreasing) in E.
2) A function is called monotone on E if it is increasing or decreasing on E.
3) If $x < y$ implies $f(x) < f(y)$ (resp. $f(x) > f(y)$) then f is said to be strictly increasing (resp. strictly decreasing) on E.

[This definition requires the ordered structure of real numbers, and thus is not applied to functions on a subset of the complex plane.]

Theorem 1.47 *(**Inverse Function Theorem**) Let f be a strictly increasing and continuous real function on $[a, b]$. Then the inverse function f^{-1} is well defined on $[f(a), f(b)]$ and is continuous.*

Proof. In this case $f(a)$ and $f(b)$ are the minimum and the maximum of f respectively, so that $f([a, b]) = [f(a), f(b)]$ [IVT: Corollary 1.43]. f is strictly monotone, so that it is one-to-one and onto mapping from $[a, b]$ to $[f(a), f(b)]$, and therefore f^{-1} exists. The continuity of f^{-1} follows from Theorem 1.44. ■
Next we study the continuity of monotone functions. For functions defined on an interval, we may talk about right-hand and left-hand limits, which however are special cases of our definition for function limits.

Definition 1.48 *1) Let f be a real or complex function in $[a, b)$ and $p \in [a, b)$. Then we say the right-hand limit of f at p exists and equals l, written as $\lim_{x \to p+} f(x) = l$ (or $\lim_{x \downarrow p} f(x) = l$), if $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s. t.*

$$|f(x) - l| < \varepsilon \quad \forall x \in [a, b) \text{ s.t. } 0 < p - x < \delta.$$

2) Let $f : (a, b] \to \mathbb{R}$ (or $\mathbb{C}$), and let $p \in (a, b]$. Then the left-hand limit of f at p exists and equals l, written as $\lim_{x \to p-} f(x) = l$, if $\forall \varepsilon > 0$, $\exists \delta > 0$ s. t.

$$|f(x) - l| < \varepsilon \quad \forall x \in (a, b] \text{ s.t.} 0 < x - p < \delta.$$

For simplicity, the left-hand limit (resp. the right-hand limit) is denoted by $f(p-)$ (resp. $f(p+)$). We will also use the notations

$$\lim_{\substack{x \to p \\ x > p}} f(x)$$

to denote the right-hand limit $f(p+)$. Similar notations apply to left-hand limits.
Obviously, $\lim_{x \to p} f(x)$ exists if and only if both the left-hand and the right-hand limits at p exist and equal.
We say f is right (or left) continuous at p if $f(p+) = f(p)$ (or $f(p-) = f(p)$) [i.e. the right-hand (or the left-hand) limit of f at p exists and equals $f(p)$]. By definition, f is continuous if and only if $f(p+) = f(p-) = f(p)$.

Example 1.49 *Consider function*

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \ ; \\ x+1 & \text{if } x < 0 \ . \end{cases}$$

Then $f(0+) = 0$ *and* $f(0-) = 1$. f *is not continuous at* 0.

Theorem 1.50 *Let* $f : (a,b) \to \mathbb{R}$ *be an increasing function. Then for every* $x_0 \in (a,b)$, *the right-hand limit and the left-hand limit of* f *at* x_0; $f(x_0+)$, $f(x_0-)$ *exist. Moreover,* $f(x_0-) = \sup_{a<x<x_0} f(x)$, $f(x_0+) = \inf_{x_0<x<b} f(x)$ *and*

$$f(x_0-) \leq f(x) \leq f(x_0+) \ .$$

The difference $f(x_0+) - f(x_0-)$ *is the "jump" of* f *at* x_0.

[There is a similar result for decreasing functions.]
Proof. By hypothesis,

$$\{f(x) : a < x < x_0\}$$

is bounded above by $f(x_0)$, and therefore has a least upper bound $\sup_{a<x<x_0} f(x)$, let us denote it by A for simplicity. Then $A \leq f(x_0)$. We have to show that $f(x_0-) = A$. Let $\varepsilon > 0$ be given. It follows from the definition of $\sup_{a<x<x_0} f(x)$, that there is a $x_\varepsilon \in (a, x_0)$ such that

$$A - \varepsilon < f(x_\varepsilon) \leq A \ .$$

Choose $\delta = x_0 - x_\varepsilon$ [which is positive]. Then, $x \in (x_\varepsilon, x_0)$ if and only if $0 < x_0 - x < \delta$, and thus, as f is increasing

$$A - \varepsilon < f(x_\varepsilon) \leq f(x) \leq A \qquad \forall 0 < x_0 - x < \delta \ .$$

By definition $f(x_0-) = \sup_{a<x<x_0} f(x)$. ■

Example 1.51 *Let* $\{c_n\}$ *be a sequence of positive numbers such that* $\sum c_n$ *converges. Let* $\{x_n\}$ *be an (arranged) countable subset of* (a,b) *[For example all rationales in* (a,b)*]. Consider*

$$f(x) = \sum_{n : x_n < x} c_n \qquad (a < x < b) \ ,$$

where the summation takes over those indices n *for which* $x_n < x$. *If there are no* $x_n < x$, *then the sum is assumed value zero.* f *is increasing on* (a,b), discontinuous *at* x_n *with an jump* $f(x_n+) - f(x_n-) = c_n$, *and is continuous at every other point of* (a,b). *Moreover* f *is a* left-continuous *at* x_n: $f(x_n-) = f(x_n)$.

Exercise 1.52 *Modify the definition of* f *in the above example so that* f *is right-continuous at each* x_n *in the sense that* $f(x_n+) = f(x_n)$.

Theorem 1.53 *If* $f : (a,b) \to \mathbb{R}$ *is increasing (or decreasing function), then* f *is continuous on* (a,b) *except* at most countable many *points.*

Proof. Suppose f is continuous in (a,b). For every $x \in (a,b)$, both side-limits $f(x-)$ and $f(x+)$ exist, and

$$f(x-) \le f(x) \le f(x+)$$

[Theorem 1.50]. Clearly f is continuous at x iff $f(x-) = f(x+)$ (i.e. the *open interval* $(f(x-), f(x+))$ is empty). If $x < y$ are two points in (a,b), then, since f is increasing,

$$f(x+) = \inf_{z>x} f(z) = \inf_{y>z>x} f(z) \le \sup_{z<y} f(z) = f(y-)$$

so that we have

$$f(x-) \le f(x) \le f(x+) \le f(y-) \le f(y) \le f(y+) \ .$$

In particular,

$$(f(x-), f(x+)) \cap (f(y-), f(y+)) = \emptyset$$

for any $x \neq y$. For any $x \in (a,b)$ at which f is discontinuous, then $(f(x-), f(x+))$ is non-empty, so that we may choose a rational number $r_x \in (f(x-), f(x+))$ [rationales are dense in $\mathbb{R}$]. r_x are different for different x, so that the set of discontinuous points of f corresponds to a subset of rationales, and thus is at most countable. ■

1.4 Infinite limits and limits at infinity

There are some variations of function limits which are quite useful as well.

Definition 1.54 *1) Let f be a real or complex function defined on $E \subset \mathbb{R}$. It is said that $f(x) \to l$ as $x \to +\infty$ (resp. $x \to -\infty$), written as $\lim_{x\to+\infty} f(x) = l$ (resp. $\lim_{x\to-\infty} f(x) = l$), if $\forall\ \varepsilon > 0$, $\exists$ a number $N > 0$ such that*

$$|f(x) - l| < \varepsilon \qquad \forall x \in E \ s.t.\ x > N \quad (resp.\ .x < -N).$$

2) Let f be a real or complex function defined on $E \subset \mathbb{C}$. Then $f(z) \to l$ as $z \to \infty$, if $\forall$ $\varepsilon > 0$, $\exists$ a number $N > 0$ such that

$$|f(z) - l| < \varepsilon \qquad \forall z \in E \ s.t.\ |z| > N \ .$$

If f is a function defined on $E \subseteq \mathbb{R}$, then, $\lim_{x\to\infty} f(x)$ means $\lim_{x\to+\infty} f(x)$ unless otherwise specified [which is thus different from $\lim_{z\to\infty} f(z)$ considering f as a function in the complex plane].

Exercise 1.55 *1. Give definitions of* $\lim_{x\to x_0} f(x) = +\infty$, $\lim_{x\to x_0} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = \infty$ *and etc.*
2. Form a statement that f does not tend to l as $x \to +\infty$.

Example 1.56 *Show that* $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to-\infty} \left(1 + \frac{1}{x}\right)^x$ *exist.*

Let $a_n = \left(1+\frac{1}{n}\right)^n$. Then

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots \\ &\quad +\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right),\end{aligned}$$

so that a_n is increasing. Moreover

$$\begin{aligned}0 &\le a_n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \\ &\le 2+\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(n-1)n} \\ &< 3.\end{aligned}$$

Hence $\{a_n\}$ is increasing and bounded, so that $\lim_{n\to\infty} a_n = \sup_n \left(1+\frac{1}{n}\right)^n$ exists. This limit is denoted by a.
If $x>0$, we use $[x]$ to denote the integer part of x. Obviously $[x] \ge x-1 \to \infty$ as $x \to \infty$. Since

$$\begin{aligned}\left(1+\frac{1}{x}\right)^x &\ge \left(1+\frac{1}{[x]+1}\right)^{[x]} \\ &= \left(1+\frac{1}{[x]+1}\right)^{[x]+1}\frac{[x]+1}{[x]+2} \to a\end{aligned}$$

and

$$\left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{[x]}\right)^{[x]+1} = \left(1+\frac{1}{[x]}\right)^{[x]}\frac{[x]+1}{[x]} \to a$$

the Sandwich Rule (or called the Squeezed Lemma) [Analysis I. You should formulate a version for function limits and prove it !] implies that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = a\ .$$

For negative x, we set $y=-x>0$. Then

$$\begin{aligned}\left(1+\frac{1}{x}\right)^x &= \left(1-\frac{1}{y}\right)^{-y} \\ &= \left(\frac{y-1}{y}\right)^{-y} = \left(\frac{y}{y-1}\right)^{y} \\ &= \left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right) \to a\ .\end{aligned}$$

[We can show that $a=e$ where $e=\sum_{n=0}^{\infty}\frac{1}{n!}$].

1.5 Uniform Convergence

Synopsis: Sequences and series of functions. The uniform limit of a sequence of continuous functions is continuous. Weierstrass's M-test for uniformly convergent series of functions. Continuity of functions defined by power series.

Let E be a subset of $\mathbb{R}$ or $\mathbb{C}$, and $f : E \to \mathbb{C}$ be continuous at $p \in E$. Then

$$\lim_{x \to p} f(x) = f(p) = f(\lim_{x \to p} x) ,$$

that is, we may interchange the function operation f and the limiting process $\lim_{x\to p}$. In many situations, we would like to understand if the *order* of performing two (or more) mathematical operations is relevant or not.

Consider a sequence $\{f_n\}$ of functions defined on E ($\subset \mathbb{R}$ or $\mathbb{C}$). If for every $x \in E$, the sequence $f_n(x) \to f(x)$, then we say that f_n converges (to f) on E, and f is the limit function, written $\lim_{n\to\infty} f_n = f$ in E or $f_n \to f$ on E. We are interested in the following question: can we exchange the order of taking two limits $\lim_{n\to\infty}$ and $\lim_{x\to p}$:

$$\lim_{x \to p} \lim_{n \to \infty} f_n(x) \quad \text{and} \quad \lim_{n \to \infty} \lim_{x \to p} f_n(x) \ ?$$

In particular, if all f_n are continuous at p, is the limit function $\lim_{n\to\infty} f_n$ continuous at p as well?

We may ask the same question for series of functions. If the sequence of partial sums

$$s_n(x) \equiv \sum_{k=1}^{n} f_k(x) \qquad \forall x \in E$$

converges for every $x \in E$, then we will use

$$\sum_{n=1}^{\infty} f_n$$

to denote the limit function of $\{s_n\}$, called the sum of the series $\sum_{n=1}^{\infty} f_n$. Can we exchange the summation $\sum_{n=1}^{\infty}$ [which by definition is understood as $\lim_{n\to\infty} \sum_{k=1}^{n}$] and $\lim_{x\to p}$:

$$\lim_{x \to p} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x \to p} f_n(x) \ ?$$

In other words, can we work out the limit $\lim_{x\to p}$ of the infinite sum $\sum_{n=1}^{\infty} f_n$ term by term?

Example 1.57 *Consider the sequence of functions [sketch their graphs!]*

$$f_n(x) = \begin{cases} 0 & \text{if } x \geq \frac{1}{n} ; \\ -nx + 1 & \text{if } 0 \leq x < \frac{1}{n} . \end{cases}$$

Then

$$\lim_{n \to \infty} f_n(x) = f(x) \equiv \begin{cases} 0 & \text{if } x \neq 0 ; \\ 1 & \text{if } x = 0 . \end{cases}$$

$f_n(x)$ converges to $f(x)$ for every $x \in [0,1]$ [but not uniformly, see definition below]. The limit function f is not continuous at 0, although all f_n are continuous on $[0,1]$. Indeed

$$\lim_{x\to 0}\lim_{n\to\infty} f_n(x) = \lim_{x\to 0} f(x) = 0$$

while

$$\lim_{n\to\infty}\lim_{x\to 0} f_n(x) = \lim_{n\to\infty} 1 = 1$$

so that

$$\lim_{x\to 0}\lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty}\lim_{x\to 0} f_n(x) \ .$$

Definition 1.58 *Let f_n be a sequence of real (or complex) functions on E.*
1) Let $f : E \to \mathbb{R}$ (or $\mathbb{C}$). If $\forall \varepsilon > 0$, $\exists N$ [a natural number depending on ε] s. t.

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall x \in E$$

whenever $n \geq N$, then we say f_n converges to f uniformly on E, written $f_n \to f$ uniformly on E (as $n \to \infty$).
2) Define the sequence of partial sums

$$s_n(x) \equiv \sum_{k=1}^{n} f_k(x) \qquad \forall x \in E$$

If $s_n \to s$ uniformly on E, then we say the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.

By definition, $f_n \to f$ uniformly on E implies point-wise convergence

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad\qquad \forall x \in E$$

Theorem 1.59 *Let f_n, $f : E \to \mathbb{R}$ (or $\mathbb{C}$). Then $f_n \to f$ uniformly on E iff*

$$\lim_{n\to\infty}\sup_{x\in E}|f_n(x) - f(x)| = 0 \ .$$

Proof. ($\Longrightarrow$) Suppose $f_n \to f$ *uniformly on* E, then for given $\varepsilon > 0$ there is N such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \text{for all } x \in E \text{ and } n > N \ .$$

[so that $\frac{\varepsilon}{2}$ is a upper bound of $\{|f_n(x) - f(x)| : x \in E\}$]. Then

$$\begin{aligned}\sup_{x\in E}|f_n(x) - f(x)| &\leq \frac{\varepsilon}{2} && [\text{Why “} \leq \text{”, not “} < \text{” ?}] \\ &< \varepsilon && \forall n > N \ .\end{aligned}$$

According to definition, $\lim_{n\to\infty}\sup_{x\in E}|f_n(x) - f(x)| = 0$.
($\Longleftarrow$) Suppose $\lim_{n\to\infty}\sup_{x\in E}|f_n(x) - f(x)| = 0$, then $\forall\varepsilon > 0$ $\exists N$ such that

$$\sup_{x\in E}|f_n(x) - f(x)| < \varepsilon \qquad\qquad \forall n > N \ .$$

Therefore

$$|f_n(x) - f(x)| \leq \sup_{x\in E}|f_n(x) - f(x)| < \varepsilon$$

for all $x \in E$ and $n > N$. That is $f_n \to f$ *uniformly on* E. ∎

Exercise 1.60 *Prove that* $f_n \to f$ *uniformly in* E *iff for any sequence* $\{x_n\}$ *in* E

$$\lim_{n\to\infty} |f_n(x_n) - f(x_n)| = 0 \ .$$

[Formulate the contrapositive to that $f_n \to f$ uniformly in E].

Theorem 1.61 ***(Cauchy's Criterion for Uniform Convergence)*** *Let* $f_n : E \to \mathbb{R}$ *(or* $\mathbb{C}$*). Then* f_n *converges uniformly on* E*, if and only if* $\forall \varepsilon > 0$*,* $\exists$ *an integer* N *such that*

$$\sup_{x\in E} |f_n(x) - f_m(x)| < \varepsilon \qquad \forall n, m > N. \tag{1.4}$$

Proof. ($\Longrightarrow$) Suppose f_n converges uniformly on E with limit function f, then $\forall\ \varepsilon > 0$, $\exists N$ s. t.

$$\sup_{x\in E} |f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \forall n > N \ .$$

Since

$$|f_n(x) - f_m(x)| \le |f_n(x) - f(x)| + |f_m(x) - f(x)|$$

so that

$$\begin{aligned} \sup_{x\in E} |f_n(x) - f_m(x)| &\le \sup_{x\in E} |f_n(x) - f(x)| + \sup_{x\in E} |f_m(x) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

as long as n, $m > N$.

($\Longleftarrow$) Conversely, suppose (1.4) holds. Then for any $x \in E$, $\{f_n(x)\}$ is a Cauchy sequence, so that it is convergent. Let us denote its limit by $f(x)$. For every $\varepsilon > 0$, choose an integer N such that

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2} \qquad \text{for all } n,\ m > N \text{ and every } x \in E.$$

For any fixed $n > N$ and $x \in E$, letting $m \to \infty$ in the above inequality we obtain

$$\begin{aligned} |f_n(x) - f(x)| &= \lim_{m\to\infty} |f_n(x) - f_m(x)| \\ &\le \frac{\varepsilon}{2} \qquad \text{[Why “} \le \text{”, not “} < \text{” ?]} \\ &< \varepsilon \ . \end{aligned}$$

According to definition, $f_n \to f$ uniformly on E. ■

Remark 1.62 (Cauchy's criterion of uniform convergence for series) *A series* $\sum_{n=1}^{\infty} f_n$ *is uniformly convergent in* E *iff* $\forall \varepsilon > 0$*,* $\exists$ *integer* N *such that*

$$\sup_{x\in E} \left| \sum_{k=m+1}^{n} f_k(x) \right| < \varepsilon \qquad \forall n > m \ge N \ .$$

[Apply Cauchy's criterion to the partial sum sequence $\{s_n\}$: $s_n = \sum_{k=1}^n f_k$].
As a consequence, we prove the following simple but important uniform convergence test for series.

Theorem 1.63 *(**Weierstrass M-Test** [for Uniform Convergence of Series]) Let $\{f_n\}$ be a sequence of (real or complex) functions defined on E. If*

$$|f_n(x)| \leq M_n \ ; \qquad \forall x \in E$$

for some non-negative constant M_n [The above inequality says M_n is an upper bound *of $|f_n|$ on E], and if $\sum_{n=1}^\infty M_n$ converges, then $\sum_{n=1}^\infty f_n$ converges uniformly on E. Moreover*

$$\left|\sum_{n=1}^\infty f_n(x)\right| \leq \sum_{n=1}^\infty |f_n(x)| \leq \sum_{n=1}^\infty M_n \qquad \forall x \in E \ .$$

Proof. The last inequality is obvious. For every $\varepsilon > 0$, there exists an integer N such that

$$\sum_{k=m+1}^n M_k < \varepsilon \qquad \forall n > m \geq N$$

[Cauchy's criterion for series]. Let $s_n = \sum_{k=1}^n f_k$ be the partial sum sequence of $\sum_{n=1}^\infty f_n$. Then for $n > m \geq N$

$$\begin{aligned} |s_n(x) - s_m(x)| &= \left|\sum_{k=m+1}^n f_k(x)\right| \\ &\leq \sum_{k=m+1}^n |f_k(x)| \qquad \text{[Triangle Inequality]} \\ &\leq \sum_{k=m+1}^n M_k \qquad \forall x \in E \ . \end{aligned}$$

That is, $|s_n - s_m|$ is bounded above by $\sum_{k=m+1}^n M_k$ and thus

$$\sup_{x \in E} |s_n(x) - s_m(x)| \leq \sum_{k=m+1}^n M_k < \varepsilon \ .$$

Therefore $\{s_n\}$ converges uniformly in E [Cauchy's criterion for uniform convergence]. ■

Example 1.64 *Let $E = [0, 1]$ and let*

$$f_n(x) = \frac{x}{1 + n^2x^2} \ .$$

Then $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in E$. Since

$$0 \leq f_n(x) = \frac{1}{2n}\frac{2nx}{1 + n^2x^2} \leq \frac{1}{2n} \to 0$$

so that $f_n \to f$ uniformly on $[0, 1]$.

Example 1.65 *Let*

$$f_n(x) = \frac{nx}{1+n^2x^2} \qquad \text{for } x \in [0,1].$$

Then $\lim_{n\to\infty} f_n(x) = 0$ *for every* $x \in [0,1]$. *While* $f_n(1/n) = 1/2$, *so that*

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \geq \frac{1}{2} \nrightarrow 0 \text{ as } n \to \infty$$

and thus f_n *converges to* 0 *but not uniformly in* $[0,1]$.

Example 1.66 $\sum_{n=0}^{\infty} x^n$ *converges to* $\frac{1}{1-x}$ *in* $(-1,1)$, *but not uniformly. [*$\sum_{n=0}^{\infty} x^n$ *converges uniformly on* $[-r,r]$ *for any* $0 < r < 1$, *see also Theorem 2.10 below].*

Indeed, $s_n(x) = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ tends to $\frac{1}{1-x}$ for any $|x| < 1$. On the other hand

$$\left| s_n(x) - \frac{1}{1-x} \right| = \frac{|x|^{n+1}}{|1-x|}$$

so that

$$\begin{aligned} \sup_{x\in(-1,1)} \left| s_n(x) - \frac{1}{1-x} \right| &\geq \frac{\left(\frac{n+1}{n+2}\right)^{n+1}}{|1 - \frac{n+1}{n+2}|} \\ &= \frac{n+2}{\left(1+\frac{1}{n+1}\right)^{n+1}} \to \infty \ . \end{aligned}$$

Hence $\sum_{n=0}^{\infty} x^n$ doesn't converge uniformly.

Theorem 1.67 *Let* f_n, $f : E \to \mathbb{R}$ *(or* $\mathbb{C}$*), and* $f_n \to f$ *uniformly in* E. *Suppose all* f_n *are continuous at* $x_0 \in E$, *then the limit function* f *is also continuous at* x_0. *Therefore*

$$\lim_{x\to x_0} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty} \lim_{x\to x_0} f_n(x) \ .$$

[The uniform limit of continuous functions is continuous.]
Proof. Given $\varepsilon > 0$, $\exists$ an integer N s.t.

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}; \qquad \forall n > N \text{ and } \forall x \in E \ .$$

Since f_{N+1} is continuous at x_0, $\exists\ \delta > 0$ (depending on x_0 and ε) such that

$$|f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\varepsilon}{3} \ ; \qquad \forall\ |x - x_0| < \delta \ .$$

Hence, if $|x - x_0| < \delta$ then [by the Triangle Inequality]

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_{N+1}(x)| + |f(x_0) - f_{N+1}(x_0)| \\ &\quad + |f_{N+1}(x) - f_{N+1}(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \ . \end{aligned}$$

By definition, f is continuous at x_0. ∎

Remark 1.68 *[Version for series] If $\sum_{n=1}^{\infty} f_n$ converges uniformly on E and every f_n is continuous at $x_0 \in E$, then*

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f_n(x_0).$$

In particular, if f_n is continuous on E for all n and $\sum_{n=1}^{\infty} f_n$ converges uniformly on E, then $\sum_{n=1}^{\infty} f_n$ is continuous on E.

Corollary 1.69 *Suppose the convergence radius of the power series $\sum_{n=1}^{\infty} a_n x^n$ is $0 < R \leq \infty$, then for every $0 \leq r < R$, $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on the closed disk $\{x : |x| \leq r\}$. Therefore, $\sum_{n=1}^{\infty} a_n x^n$ is continuous on the open ball $\{x : |x| < R\}$.*

Proof. According to the definition of convergence radius, $\sum_{n=1}^{\infty} a_n x^n$ is absolutely convergent for $|x| < R$. In particular, $\sum_{n=1}^{\infty} |a_n| r^n$ is convergent. Since

$$|a_n x^n| \leq |a_n| r^n \quad \text{for all } x \text{ such that } |x| \leq r$$

therefore, by Weierstrass M-test, $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $\{x : |x| \leq r\}$. ∎
The end points R and $-R$ need to be handled differently.

Theorem 1.70 *(Abel's theorem) If the series $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, 1]$. Therefore, $\sum_{n=0}^{\infty} a_n x^n$ is continuous on $(-1, 1]$, and*

$$\lim_{x \uparrow 1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n \ .$$

Proof. For every $\varepsilon > 0$, there is N such that, for every $n > m > N$ we have

$$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon \ . \tag{1.5}$$

Fix $m > N$, set

$$c_k = \sum_{j=m}^{k} a_j \quad \text{for} \quad k \geq m, \quad c_{m-1} = 0 \ .$$

Then (1.5) implies that $|c_k| < \varepsilon$ whenever $k \geq m - 1$, and $a_k = c_k - c_{k-1}$. We have

$$\begin{aligned} \sum_{k=m}^{n} a_k x^k &= \sum_{k=m}^{n} (c_k - c_{k-1}) x^k \\ &= \sum_{k=m}^{n} c_k x^k - \sum_{k=m}^{n} c_{k-1} x^k \\ &= \sum_{k=m}^{n-1} c_k \left(x^k - x^{k+1} \right) + c_n x^n \end{aligned}$$

[which is called the Abel's summation formula]. Hence, for any $x \in [0,1]$,

$$\begin{aligned}\left|\sum_{k=m}^{n} a_k x^k\right| &\leq \sum_{k=m}^{n-1} |c_k| \left(x^k - x^{k+1}\right) + |c_n| x^n \\ &< \varepsilon \sum_{k=m}^{n-1} \left(x^k - x^{k+1}\right) + \varepsilon x^n \\ &= \varepsilon x^m \leq \varepsilon \ .\end{aligned}$$

According to definition, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0,1]$. Therefore $\sum_{n=0}^{\infty} a_n x^n$ continuous on $[0,1]$. In particular

$$\lim_{x \uparrow 1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n \ .$$

■

Theorem 1.71 ***(The Dini Theorem)***. *Let f_n be a sequence of real continuous functions on $[a,b]$. Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ for any $x \in [a,b]$, where f is a continuous function on $[a,b]$, and suppose that*

$$f_n(x) \geq f_{n+1}(x) \qquad \forall\ n \text{ and } \forall x \in [a,b] \ ,$$

then $f_n \to f$ uniformly in $[a,b]$.

Proof. Let $g_n(x) = f_n(x) - f(x)$. Then g_n is continuous for every n, $g_n \geq 0$ and $\lim_{n\to\infty} g_n(x) = 0$ for any $x \in [a,b]$. Suppose $\{g_n\}$ were not uniformly convergent on $[a,b]$. Then $\exists\ \varepsilon > 0$, such that for each k there are a natural number $n_k > k$ and a point $x_k \in [a,b]$ such that

$$|g_{n_k}(x_k)| = g_{n_k}(x_k) \geq \varepsilon \ .$$

[*Contrapositive to that $\{g_n\}$ converges to 0 uniformly on $[a,b]$*]. We may choose n_k so that $k \to n_k$ is increasing, and we may assume that $x_k \to p$. [Otherwise we may argue with a convergent subsequence of $\{x_k\}$, according to Bolzano-Weierstrass' Theorem]. Then $p \in [a,b]$. For any (fixed) k, since $\{g_n\}$ is decreasing,

$$\varepsilon \leq g_{n_l}(x_l) \leq g_{n_k}(x_l) \tag{1.6}$$

for all $l > k$. Letting $l \to \infty$ in the above inequality, we obtain

$$\varepsilon \leq \lim_{l\to\infty} g_{n_k}(x_l) = g_{n_k}(p) \qquad [g_{n_k} \text{ is continuous at } p],$$

which contradicts to the assumption that $\lim_{k\to\infty} g_{n_k}(p) = 0$. ■

Example 1.72 *Let $f_n(x) = \frac{1}{1+nx}$ for $x \in (0,1)$. Then $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in (0,1)$, f_n is decreasing in n, but f_n does not converge uniformly. Dini's theorem doesn't apply, as $(0,1)$ is not compact.*

The proofs of the following theorems will be given in the Trinity term.

Theorem 1.73 *If $f_n \to f$ uniformly in $[a,b]$ and if every f_n is continuous, then*

$$\int_a^b f = \int_a^b \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_a^b f_n \ .$$

Similarly, if the series $\sum_{n=1}^{\infty} f_n$ converges uniformly in $[a,b]$ and if all f_n are continuous, then we may integrate the series term by term

$$\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n \ .$$

However, the uniform convergence is not *the right condition* for integrating a series term by term: we can exchange the order of integration $\int_a^b$ (which involves a limiting procedure) and $\lim_{n\to\infty}$ under much weaker conditions. The search for correct conditions for term-by-term integration was led to the discovery of Lebesgue's integration [Part A option: Integration]. For details, see W. Rudin's Principles, Chapter 11 (page 300).

Theorem 1.74 *Let $f_n \to f$ in (a,b) (convergence point-wisely). Suppose f_n' exists and is continuous on (a,b) for every n, and if $f_n' \to g$ uniformly in (a,b). Then f' exists and is continuous in (a,b), and*

$$\frac{d}{dx} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{d}{dx} f_n(x) \ .$$

Similarly, if $\sum f_n$ converges in (a,b), if every f_n' exists and is continuous in (a,b), and if $\sum f_n'$ converges uniformly in (a,b), then

$$\frac{d}{dx} \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} f_n' \ .$$

Chapter 2

Differentiability

Synopsis: Definition of the derivative of a function of a real variable. Algebra of derivatives, examples to include polynomials and inverse functions. Theorem that the derivative of a function defined by a power series is given by the derived series.
Vanishing of the derivative at a local maximum or minimum. Rolle's Theorem. Mean Value Theorem with simple applications: constant and monotone functions.
Cauchy's (generalized) Mean Value Theorem and l'Hôpital's formula. Taylor's Theorem with remainder in Lagrange's form; examples of Taylor's Theorem to include the binomial expansion with arbitrary index.

2.1 Differentiability

In this course we only study differentiability for real (or complex)-valued functions on E, where E is a subset of the real line $\mathbb{R}$. The differentiability for complex functions on the complex plane $\mathbb{C}$ is totally different from the real case here: the existence of complex coordinates or the complex structure has a completely different meaning, so that it requires another theory – Complex Analysis [**Part A: Analysis**].

Definition 2.1 *1) Let $f:(a,b)\to\mathbb{R}$ (or $\mathbb{C}$), and $x_0\in(a,b)$. If*

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

exists (a real or complex number), then the above limit is called the derivative of f at x_0 and is denoted by $f'(x_0)$ or $\frac{df}{dx}(x_0)$.
2) If $f:(a,b]\to\mathbb{R}$ (or $\mathbb{C}$), and $x_0\in(a,b]$, then the left-derivative of f at x_0 is defined by

$$f'(x_0-)=\lim_{x\uparrow x_0}\frac{f(x)-f(x_0)}{x-x_0}\quad \textit{[if the limit exists].}$$

If $f:[a,b)\to\mathbb{R}$ (or $\mathbb{C}$), and $x_0\in[a,b)$, then the right-derivative of f at x_0 is defined by

$$f'(x_0+)=\lim_{x\downarrow x_0}\frac{f(x)-f(x_0)}{x-x_0}\quad \textit{[if the limit exists].}$$

3) If $f : D \to \mathbb{C}$ where $D \subset \mathbb{C}$ is a domain, $z_0 \in D$, the complex derivative of f at z is defined to be

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \qquad \text{[if the limit exists]}.$$

Remark 2.2 *Let $y = f(x)$. There are other notations for derivatives*
$\frac{dy}{dx}$ or $\frac{df(x_0)}{dx}$ [G. W. Leibnitz]
y' or $f'(x_0)$ [J. L. Lagrange]
Dy or $Df(x_0)$ [A. L. Cauchy, in particular for vector-valued functions of several variables].

Remark 2.3 *According to definition, $f'(x_0)$ exists iff both side derivatives $f'(x_0-)$ and $f'(x_0+)$ exist, and $f'(x_0-) = f'(x_0+)$. If $f : (a, b) \to \mathbb{C}$, and $f'(x_0)$ exists, then we say f is differentiable.*
f is differentiable on (a, b) if it is differentiable at every point in (a, b).
We say f is differentiable on $[a, b]$ if it is differentiable on (a, b) and both $f'(a+)$ and $f'(b-)$ exist.

Remark 2.4 *We have abused the notations $f'(x_0+)$ and $f'(x_0-)$. Recall that if g is a function defined in (a, b) and $x_0 \in (a, b)$, then $g(x_0+)$ and $g(x_0-)$ represent the right-hand limit and the left-hand limit of g at x_0:*

$$g(x_0+) = \lim_{x\downarrow x_0} g(x) \qquad \text{and} \qquad g(x_0-) = \lim_{x\uparrow x_0} g(x) \ ,$$

respectively. According to definition here, even f is differentiable in (a, b) [so that the derivative function f' of f is a well-defined on (a, b)], $f'(x_0+)$ and $f'(x_0-)$ do not mean the right-hand and the left-hand limits of the Derivative Function f' at x_0! However, we will show *that,* if $\lim_{x\downarrow x_0} f'(x)$ exists, then the right-hand limit of f'; $\lim_{x\downarrow x_0} f'(x)$; does coincide with $f'(x_0+)$ *we have defined here. A similar statement holds for $f'(x_0-)$ as well. Here is a simple example to show the difference. Consider $f(x) = x^2 \sin\frac{1}{x}$ for $x \neq 0$, and $f(0) = 0$. Then we can show that $f'(0) = 0$ and*

$$f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x} \qquad \text{for } x \neq 0.$$

*Therefore $f'(0+) = f'(0-) = f'(0) = 0$, but the right-hand and left-hand limits of f at 0: $\lim_{x\downarrow 0} f'(x)$ and $\lim_{x\uparrow 0} f'(x)$ do not exist! [*You'd better to remember the function $f(x) = x^\alpha \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$, which has a value if you want to construct examples related with differentiability.*]*

Exercise 2.5 *1) If $f'(x_0-) > 0$ (resp. $f'(x_0-) < 0$), then there is a number $\delta > 0$ such that $f(x) \le f(x_0)$ (resp. $f(x) \ge f(x_0)$) for $x \in (x_0 - \delta, x_0]$.*
2) If $f'(x_0+) > 0$ (resp. $f'(x_0+) < 0$), then there is $\delta > 0$ such that $f(x) \ge f(x_0)$ (resp. $f(x) \le f(x_0)$) for $x \in [x_0, x_0 + \delta)$.
3) If $f'(x_0) > 0$ (resp. $f'(x_0) < 0$), then there is $\delta > 0$ such that

$$(f(x) - f(x_0))(x - x_0) \ge 0$$

(resp.

$$(f(x) - f(x_0))(x - x_0) \le 0 \ )$$

for $x \in (x_0 - \delta, x_0 + \delta)$.

If f is differentiable at x_0, then the increment of f near x_0

$$f(x) - f(x_0) = f'(x_0)\,(x - x_0) + o(x, x_0)$$

where o is a function of x and x_0 satisfying that

$$\lim_{x \to x_0} \frac{o(x, x_0)}{x - x_0} = 0 \ .$$

The linear part of the increment $f(x) - f(x_0)$; $f'(x_0)\,(x - x_0)$; is called *the differential* of f at x_0.
We next prove several standard facts about differentiability.

Theorem 2.6 *Let $f : (a, b) \to \mathbb{R}$ (or $\mathbb{C}$). If f is differentiable at $x_0 \in (a, b)$, then f is continuous at x_0.*

Proof. Since

$$\begin{aligned} \lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0) \\ &= f'(x_0) \times 0 \\ &= 0 \end{aligned}$$

the second equality follows from the algebra of limits. Therefore $\lim_{x \to x_0} f(x) = f(x_0)$, according to definition, f is continuous at x_0. ■

Theorem 2.7 *If f, $g : (a, b) \to \mathbb{R}$ (or $\mathbb{C}$) are differentiable at $x_0 \in (a, b)$, then*
1) $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$;
2) (Product rule) $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$ [The operation $f \to f'$ is a derivation.*]*
3) and if in addition $g(x_0) \neq 0$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)} \ .$$

Proof. 1) is clearly true. 2) Let $h = fg$. Then

$$h(x) - h(x_0) = g(x_0)\,(f(x) - f(x_0)) + f(x)\,(g(x) - g(x_0))\,,$$

dividing both sides by $x - x_0$, and taking limit $x \to x_0$ we obtain

$$\begin{aligned} \lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} &= g(x_0) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} f(x) \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= f'(x_0)g(x_0) + f(x_0)g'(x_0) \ \text{[Algebra of limits]} \end{aligned}$$

where we have used the fact that $g(x) = g(x_0)$ as $x \to x_0$ [Theorem 2.6].

To prove 3), we first need to show f/g is well-defined near x_0. Since g is continuous at x_0, for $\varepsilon = \frac{|g(x_0)|}{2}$ we can find a number $\delta > 0$ such that

$$|g(x) - g(x_0)| < \frac{|g(x_0)|}{2} \qquad \forall x \in (a,b) \text{ s.t. } |x - x_0| < \delta \ .$$

It follows that

$$\begin{aligned} |g(x)| &\geq |g(x_0)| - |g(x) - g(x_0)| \qquad \text{[Triangle Inequality]} \\ &> \frac{|g(x_0)|}{2} \qquad \forall x \in (a,b) \text{ s.t. } |x - x_0| < \delta \ . \end{aligned}$$

Let $h = \frac{f}{g}$ on $(a,b) \cap (x_0 - \delta, x_0 + \delta)$. Then

$$\frac{h(x) - h(x_0)}{x - x_0} = \frac{1}{g(x)g(x_0)} \left[g(x_0) \frac{f(x) - f(x_0)}{x - x_0} - f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right] \ .$$

Letting $x \to x_0$ we prove 3). ■

Theorem 2.8 *(The Chain Rule) Suppose $f : (a,b) \to \mathbb{R}$ is differentiable at $x_0 \in (a,b)$, $g : (c,d) \to \mathbb{R}$ is differentiable at $y_0 = f(x_0) \in (c,d)$, and $f((a,b)) \subseteq (c,d)$, then $h = g \circ f$ is differentiable at x_0 and*

$$h'(x_0) = g'(y_0) f'(x_0) \ .$$

Proof. Let

$$v(y) = \frac{g(y) - g(y_0)}{y - y_0} - g'(y_0) \qquad \forall y \neq y_0$$

and $v(y_0) = 0$. Then, $v(y) \to 0$ as $y \to y_0$, so that v is continuous at y_0. We can then write the increment

$$g(y) - g(y_0) = (y - y_0)\left(g'(y_0) + v(y)\right)$$

which is valid for any $y \in (c,d)$. In particular

$$g(f(x)) - g(f(x_0)) = (f(x) - f(x_0))\left(g'(y_0) + v(f(x))\right)$$

so that

$$\frac{h(x) - h(x_0)}{x - x_0} = g'(y_0) \frac{f(x) - f(x_0)}{x - x_0} + v(f(x)) \frac{f(x) - f(x_0)}{x - x_0} \ . \tag{2.1}$$

Since f is differentiable at x_0, f continuous at x_0 [Theorem 2.6]. Therefore $f(x) \to y_0$ as $x \to x_0$, which then yields that $v(f(x)) \to 0$ as $x \to x_0$. Letting $x \to x_0$ in (2.1) we obtain

$$\begin{aligned} \lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} &= g'(y_0) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &\quad + \lim_{x \to x_0} v(f(x)) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= g'(y_0) f'(x_0) + 0 \times f'(x_0) \\ &= f'(x_0) g'(y_0) \ . \end{aligned}$$

■

Theorem 2.9 *Suppose f is a differentiable real function on (a,b), with $f' \neq 0$, and suppose f is strictly increasing in (a,b). Write $y = f(x)$. Then the inverse function $g = f^{-1} : f((a,b)) \to \mathbb{R}$ is differentiable, and*

$$g'(y) = \frac{1}{f'(f^{-1}(y))} \qquad \forall y \in f((a,b)) \ .$$

Proof. $g = f^{-1}$ is continuous on $f((a,b))$ [Apply Inverse Function Theorem to any $[c,d] \subset (a,b)$, Chapter 1]. Let $y_0 \in f((a,b))$ and $x_0 \in (a,b)$ such that $f(x_0) = y_0$. Then

$$\begin{aligned}
\lim_{y\to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} &= \lim_{y\to y_0} \frac{x - x_0}{f(x) - f(x_0)} \\
&= \lim_{y\to y_0} \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}
\end{aligned}$$

where $f(x) = y$. Since f^{-1} is continuous, $x \to x_0$ as $y \to y_0$. Hence

$$\frac{f(x) - f(x_0)}{x - x_0} \to f'(x_0) \quad \text{as } \ y \to y_0$$

so that

$$\lim_{y\to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}$$

exists [Algebra of limits is applicable as $f'(x_0) \neq 0$]. $g = f^{-1}$ is differentiable at y_0, and

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))} \ .$$

■

Theorem 2.10 *Consider the power series*

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n z^n \\
&= a_0 + a_1 z + \cdots + a_n z^n + \cdots \ .
\end{aligned} \tag{2.2}$$

Let R be its convergence radius, where $0 < R \leq +\infty$. Then
1) The power series obtained by differentiating f term by term

$$\begin{aligned}
g(z) &= \sum_{n=1}^{\infty} n a_n z^{n-1} \\
&= a_1 + 2a_2 z \cdots + n a_n z^{n-1} + \cdots \ .
\end{aligned} \tag{2.3}$$

has the convergence radius R. In particular for any $0 \leq r < R$

$$\sum_{n=1}^{\infty} n|a_n|r^{n-1} < +\infty \qquad \text{[Absolute Convergence at } z = r\text{]} \ . \tag{2.4}$$

2) The [complex] derivative

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists for every z satisfying $|z| < R$, and $f'(z) = g(z)$. That is

$$\frac{d}{dz} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad \forall |z| < R \ . \tag{2.5}$$

[We may differentiate a power series term by term]

Proof. 1) Let $|z| < R$. Set $r = \frac{1}{2}(|z| + R)$ (or $r = 2|z| + 1$ if $R = +\infty$). Then $|z| < r < R$, $q = \frac{|z|}{r} < 1$, and we have the following facts:

(a) $\sum_{n=0}^{\infty} |a_n| r^n < +\infty$ [Analysis 1: a power series converges absolutely inside its convergence disk],

(b) $\{nq^{n-1}\}$ is bounded [Indeed $\sum nq^{n-1}$ converges (by ration test), so that $\lim_{n\to\infty} nq^{n-1} = 0$: but we don't need these stronger results here]. Let $b_n = nq^{n-1}$. Then

$$\frac{b_{n+1}}{b_n} = \frac{n+1}{n} q$$

which is smaller then 1 for n big enough, so that $\{b_n\}$ is decreasing for big n, so that $\lim_{n\to\infty} b_n$ exists, hence $\{nq^{n-1}\}$ is bounded: $nq^{n-1} \le M$ for some M, $\forall n$.

(c) $\sum_{n=1}^{\infty} n a_n z^{n-1}$converges absolutely. Indeed

$$\begin{aligned} |n a_n z^{n-1}| &\le n|a_n||z|^{n-1} = nq^{n-1}|a_n| r^{n-1} \\ &\le \frac{M}{r} |a_n| r^n \qquad \forall n \ge 1 \end{aligned}$$

so that, by the comparison test [Analysis 1]

$$\sum_{n=1}^{\infty} n|a_n||z|^{n-1} \le \frac{M}{r} \sum_{n=1}^{\infty} |a_n| r^n < +\infty$$

Similarly we may prove the convergence radius of $\sum_{n=1}^{\infty} n a_n z^{n-1}$ can not be greater then that of $\sum_{n=0}^{\infty} a_n z^n$.

2) We are going to show that the complex derivative of f at any point z such that $|z| < R$. Let $r = \frac{1}{2}(|z| + R)$ (or $r = |z| + 1$ if $R = \infty$). Then $r < R$, and $|z| < r$. For any point w such that $|w| < r$, consider

$$\begin{aligned} \frac{f(w) - f(z)}{w - z} - g(z) &= \sum_{n=1}^{\infty} a_n \left(\frac{w^n - z^n}{w - z} - n z^{n-1} \right) \\ &= \sum_{n=2}^{\infty} a_n \left(\frac{w^n - z^n}{w - z} - n z^{n-1} \right) \ ; \end{aligned} \tag{2.6}$$

where we have added the series $f(w)$, $f(z)$ and $g(z)$ term by term, which is justified as all these series are absolutely convergent [Analysis 1: a power series converges absolutely inside the convergence disk]. Our aim is to show that

$$\frac{f(w) - f(z)}{w - z} - g(z) \to 0 \qquad \text{as } w \to z \ .$$

To this end we use the identity

$$\frac{w^n - z^n}{w - z} = z^{n-1} + z^{n-2}w + \cdots + zw^{n-2} + w^{n-1}$$

[*Exercise.* Apply the geometric series

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x} \qquad \forall n \geq 1$$

to $x = w/z$ or z/w]. Therefore, for any $w \neq z$ and $n \geq 2$

$$\begin{aligned}
\frac{w^n - z^n}{w - z} - nz^{n-1} &= z^{n-1} + z^{n-2}w + \cdots + zw^{n-2} + w^{n-1} \\
&\quad -z^{n-1} - z^{n-1} - \cdots - z^{n-1} - z^{n-1} \\
&= \sum_{k=1}^{n-1} \left(z^{n-1-k} w^k - z^{n-1} \right) \\
&= \sum_{k=1}^{n-1} z^{n-1-k} \left(w^k - z^k \right) \ .
\end{aligned}$$

Let

$$h_n(w) = a_n \sum_{k=1}^{n-1} z^{n-1-k} \left(w^k - z^k \right) \ ; \quad n = 2, 3, \cdots .$$

Then

$$\frac{f(w) - f(z)}{w - z} - g(z) = \sum_{n=2}^{\infty} h_n(w)$$

All h_n are continuous in $\mathbb{C}$ (polynomials in w), and $h_n(z) = 0$ ($\forall n \geq 2$). We claim that $\sum_{n=2}^{\infty} h_n(w)$ converges uniformly in $|w| \leq r$. In fact

$$\begin{aligned}
|h_n(w)| &\leq |a_n| \sum_{k=1}^{n-1} |z|^{n-1-k} \left(|w|^k + |z|^k \right) \\
&\leq 2n|a_n| r^{n-1} \ .
\end{aligned}$$

By 1), $\sum n|a_n| r^{n-1} < +\infty$, so that $\sum_{n=2}^{\infty} h_n(w)$ converges uniformly in closed disk $\{w : |w| \leq r\}$ [Weierstrass M-test, Chapter 2]. Hence $\sum_{n=2}^{\infty} h_n(w)$ is continuous in the disk $|w| \leq r$ [Theorem 1.67: the uniform limit of continuous functions is continuous]. Therefore

$$\lim_{w \to z} \sum_{n=2}^{\infty} h_n(w) = \sum_{n=2}^{\infty} h_n(z) = 0$$

so that

$$\begin{aligned}
\lim_{w \to z} \frac{f(w) - f(z)}{w - z} &= \lim_{w \to z} \left(\frac{f(w) - f(z)}{w - z} - g(z) \right) + g(z) \\
&= \lim_{w \to z} \sum_{n=2}^{\infty} h_n(w) + g(z) \\
&= g(z) \ .
\end{aligned}$$

■

Proposition 2.11 *1) Define* $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ *[*$0! = 1$ *and the convergence radius is* ∞*]. Then* $\frac{d}{dz}\exp(z) = \exp(z)$*.*
2) Define $\sin(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$ *and* $\cos(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$ *, both are differentiable in* $\mathbb{C}$*, and* $\frac{d}{dz}\sin(z) = \cos(z)$ *and* $\frac{d}{dz}\cos(z) = -\sin(z)$*.*

Then exp is differentiable in $\mathbb{C}$ [Theorem 2.10] and

$$\begin{aligned} \frac{d}{dz}\exp(z) &= \sum_{n=1}^{\infty} n\frac{z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z) \ . \end{aligned}$$

Similarly

$$\begin{aligned} \frac{d}{dz}\sin(z) &= \sum_{n=0}^{\infty}(-1)^n(2n+1)\frac{z^{2n}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} = \cos(z) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dz}\cos(z) &= \sum_{n=1}^{\infty}(-1)^n 2n\frac{z^{2n-1}}{(2n)!} \\ &= -\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!} = -\sin(z) \ . \end{aligned}$$

Example 2.12 *Clearly* $\exp(0) = 1$ *and* $\exp(x) \geq 1$ *for every* $x \geq 0$*. Moreover* $x \to \sum_{n=0}^{\infty} \frac{x^n}{n!}$ *on* $[0, +\infty)$ *is strictly increasing [Indeed it is strictly increasing on* $(-\infty, \infty)$*, see below Corollary 2.3]. Let* $f(x) = \exp(x)$ $(x \geq 0)$*, and use* $\log : [1, +\infty) \to [0, +\infty)$ *to denote the inverse function of* f*. Then* $\log$ *is differentiable, and [Theorem 2.9]*

$$\frac{d}{dy}\log y = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f(f^{-1}(y))} = \frac{1}{y} \ ,$$

that is, $\frac{d}{dx}\log x = \frac{1}{x}$ *(for all* $x \geq 1$*).*

Example 2.13 *Consider function*

$$f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if} \quad x \neq 0 \ ; \\ 0 & \text{if} \quad x = 0 \ , \end{cases}$$

which is continuous on $\mathbb{R}$*. Since*

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\sin\frac{1}{x}$$

doesn't exist, f *is not differentiable at* 0*.* f *is differentiable at any other point, and*

$$f'(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} \qquad \forall x \neq 0 \ .$$

Note $\lim_{x\to 0} f'(x)$ *does not exist [Why ?]*

Example 2.14 *Let* $f(x) = x^2 \sin \frac{1}{x}$ *$(x \neq 0)$ and* $f(0) = 0$. *Then*

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x} \\ &= \lim_{x\to 0} x \sin \frac{1}{x} = 0 \end{aligned}$$

and

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \qquad \forall x \neq 0 \ .$$

Therefore f is differentiable everywhere, the derivative function f' is not continuous at 0: $\lim_{x\to 0} f'(x)$ doesn't exist.

Example 2.15 *$f(x) = |x|$ is continuous but not differentiable at 0. But the left (right)-derivative of f at 0 exists, and $f'(0-) = -1$ and $f'(0+) = 1$. Note that $\lim_{x\downarrow 0} f'(x) = f'(0+)$ and $\lim_{x\to\uparrow 0} f'(x) = f'(0-)$.*

Example 2.16 ***(B. L. Van der Waerden)*** *[I don't think I'll have time to work through this example] [A function which is continuous on $\mathbb{R}$ but nowhere differentiable]. Let*

$$h(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1; \\ 2 - x & \text{if } 1 \leq x \leq 2 \end{cases}$$

and extend h to be a periodic function with period 2, i.e. $h(x+2) = h(x)$ for $x \in \mathbb{R}$. Then h is continuous on $\mathbb{R}$. Consider the series

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n h(4^n x) \ .$$

By the Weierstrass M-test, $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n h(4^n x)$ converges uniformly in $\mathbb{R}$, and thus f is continuous on $\mathbb{R}$ [Theorem 1.67] and

$$|f(x)| \leq \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = 4 \qquad \text{for every } x \in \mathbb{R} \ .$$

Let $x \in \mathbb{R}$, $m \in \mathbb{N}$ and set $k = [4^m x]$ the integer part of $4^m x$: k is the unique integer such that

$$k \leq 4^m x < k + 1 \ .$$

Let $\alpha_m = 4^{-m} k$ and $\beta_m = 4^{-m}(k+1)$. Obviously

$$\alpha_m \leq x < \beta_m$$

and

$$\beta_m - \alpha_m = \frac{1}{4^m} \to 0 \quad \text{as } m \to \infty \ .$$

In particular, $\lim_{m\to\infty} \alpha_m = \lim_{m\to\infty} \beta_m = x$. We are going to show that

$$\lim_{m\to\infty} \frac{f(\beta_m) - f(\alpha_m)}{\beta_m - \alpha_m}$$

does not exist, so that f is not differentiable at x. Since x is arbitrary, thus f is nowhere differentiable.
If $n > m$, then $4^n\beta_m - 4^n\alpha_m$ is an even number, and if $n \leq m$ then there is no integer between $4^n\beta_m$ and $4^n\alpha_m$. Therefore

$$|h(4^n\beta_m) - h(4^n\alpha_m)| = \begin{cases} 0, & \text{if } n > m; \\ 4^{n-m}, & \text{if } n \leq m. \end{cases}$$

Hence

$$\begin{aligned} f(\beta_m) - f(\alpha_m) &= \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n (h(4^n\beta_m) - h(4^n\alpha_m)) \\ &= \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n (h(4^n\beta_m) - h(4^n\alpha_m)) \end{aligned}$$

so that

$$\begin{aligned} |f(\beta_m) - f(\alpha_m)| &\geq \left(\frac{3}{4}\right)^m - \sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n |h(4^n\beta_m) - h(4^n\alpha_m)| \\ &= \left(\frac{3}{4}\right)^m - \sum_{n=0}^{m-1} 4^{n-m} \left(\frac{3}{4}\right)^n \\ &= \left(\frac{3}{4}\right)^m - \frac{1}{4^m}\frac{3^m - 1}{2} \\ &= \frac{1}{2}\left(\frac{3}{4}\right)^m + \frac{1}{2}\frac{1}{4^m} . \end{aligned}$$

Therefore

$$\frac{|f(\beta_m) - f(\alpha_m)|}{\beta_m - \alpha_m} \geq \frac{3^m + 1}{2} \longrightarrow +\infty \quad \text{as } m \to \infty$$

and it follows thus that $\lim \frac{f(\beta_m)-f(\alpha_m)}{\beta_m-\alpha_m}$ does not exist, so that f is not differentiable at any point x.

Definition 2.17 *If f is differentiable on (a, b), then the second-order derivative*

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

if the limit exists, which is denoted also by $f^{(2)}(x)$. Inductively define $f^{(n+1)}(x)$ to be the derivative $f^{(n)}$ for any n, as long as the derivative exists.

Theorem 2.18 ***(Leibnitz Formula)*** *If $F = fg$, then*

$$F^{(n)}(x) = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(x) g^{(n-j)}(x) .$$

2.2 The Mean Value Theorem (MVT)

2.2.1 Local maxima and minima

If $f : E \to \mathbb{R}$ is a real function on E, then $x_0 \in E$ is a local maximum (resp. local minimum) if there is a $\delta > 0$ such that

$$f(x) \le f(x_0) \quad (\text{resp.} \quad f(x) \ge f(x_0))$$

whenever $x \in (x_0-\delta, x_0+\delta)\cap E$. A local maximum or minimum is called a local extremum.

Theorem 2.19 ***(Fermat's Theorem)*** *Let $f : E \to \mathbb{R}$. Suppose x_0 is a local extremum of f, $(x_0 - \varepsilon, x_0 + \varepsilon) \subset E$ for some $\varepsilon > 0$, and f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof. Let us prove Fermat's theorem for a local maximum x_0. Then $f(x) - f(x_0) \le 0$ whenever $x - x_0 > 0$ and $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$, so that

$$f'(x_0+) = \lim_{x\to x_0+} \frac{f(x) - f(x_0)}{x - x_0} \le 0$$

On the other hand, for $x \in (x_0 - \varepsilon, x_0)$, $f(x) - f(x_0) \le 0$ and $x - x_0 < 0$, hence

$$f'(x_0-) = \lim_{x\to x_0-} \frac{f(x) - f(x_0)}{x - x_0} \ge 0 .$$

Since f is differentiable at x_0, we therefore must have $f'(x_0) = f'(x_0-) = f'(x_0+) = 0$. ■

Theorem 2.20 ***(Rolle's Theorem, 1691)*** *If $f : [a, b] \to \mathbb{R}$ is continuous, f is differentiable on (a, b) and $f(a) = f(b)$, then there exists a point $x_0 \in (a, b)$ such that $f'(x_0) = 0$.*

Proof. If f is constant in $[a, b]$, then $f'(x) = 0$ for every $x \in (a, b)$, so that any point $x_0 \in (a, b)$ will do. Since f is continuous, f attains its maximum and minimum on $[a, b]$. That is, there are two points x_1, $x_2 \in [a, b]$ such that $f(x_1) = \min_{x\in[a,b]} f(x)$ and $f(x_2) = \sup_{x\in[a,b]} f(x)$. If f is not constant, then $f(x_1) \ne f(x_2)$. Since $f(a) = f(b)$, at least one (say x_0) of x_1 and x_2 belongs to (a, b). x_0 must be a local extremum and therefore [by Fermat's Theorem] $f'(x_0) = 0$. ■

Corollary 2.21 *Suppose $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then between any two distinct roots of $f(x) = 0$ there is a root of $f'(x) = 0$.*

Example 2.22 $f(x) = \sin x$ *and* $f'(x) = \cos x$. *Study the zeros of f and f'.*

2.2.2 Mean Value Theorems

[One of the most important results in this course]

Theorem 2.23 ***(Mean Value Theorem, MVT)*** *If $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b), then there is a point $x_0 \in (a, b)$ such that*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a} .$$

[In applications, we often write MVT as

$$f(b) - f(a) = f'(x_0)\,(b-a)$$

for some x_0 which depends on a and b.]

Proof. Apply Rolle's theorem to function

$$F(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right] .$$

Then $F'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, and $F(a) = 0 = F(b)$. By Rolle's Theorem, $F'(\xi) = 0$ for some number ξ between a and b. That is, $f'(\xi) = \frac{f(b)-f(a)}{b-a}$. ■

Corollary 2.24 ***[Identity Theorem]*** *If $f : (a,b) \to \mathbb{R}$ is differentiable and $f' = 0$, then f is constant on (a,b).*

Proof. Apply MVT to f on $[x,y]$ where x, y are any two points in (a,b). Then $f(x) - f(y) = f'(\xi)(x-y)$ for some number ξ between x and y. Since $f'(\xi) = 0$, so that $f(x) = f(y)$. Therefore f is constant in (a,b). ■

Corollary 2.25 *Let $f : (a,b) \to \mathbb{R}$ be differentiable.*
1) If $f'(x) \geq 0$ for every $x \in (a,b)$ then f is increasing.
2) If $f'(x) \leq 0$ for every $x \in (a,b)$ then f is decreasing.

Recall what we did last Friday. The main idea is to look at those interesting points at which the graph of a continuous function turns: one class of such points are local maxima and local minima.

1. **Fermat's Theorem** If f is continuous in (a,b) and x_0 is a local maximum or local minimum of f in (a,b), then $f'(x_0) = 0$. [You should be able to prove this theorem: the idea is to figure out the signs of $f'(x_0-)$ and $f'(x_0+)$.]

2. **Rolle's Theorem**. If $f : [a,b] \to \mathbb{R}$ is (1) continuous, (2) differentiable in (a,b) and (3) $f(a) = f(b)$, then there is (at least) a point ξ between a and b such that $f'(\xi) = 0$. [It follows directly that a) f attains its bounds and b) Fermat's Theorem].

3. **Corollary**. Suppose $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then between any two distinct roots of $f(x) = 0$ there is a root of $f'(x) = 0$.

4. **Mean Value Theorem**. If $f : [a,b] \to \mathbb{R}$ is (1) continuous, (2) differentiable in (a,b), then there is (at least) one point ξ between a and b such that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$. [You should be able to prove this very important theorem. The proof uses the Rolle Theorem. On the other hand, Rolle's Theorem is the special case of MVT. Therefore MVT and Rolle's Theorem are equivalent !]

We may state the MVT as the following

$$f(b) - f(a) = f'(\xi)(b - a)$$

for some $\xi \in (a, b)$. ξ can be written as $\xi = a + \theta(b - a)$ where $\theta \in (0, 1)$. Therefore, if we set $h = b - a$, then $b = a + h$, so that the MVT becomes

$$f(a + h) - f(a) = f'(a + \theta h)h$$

or in the form:

$$f(a + h) = f(a) + f'(a + \theta h)h$$

[which is the special of Taylor's Theorem].

Remark 2.26 *Recall the following conditions we used in these theorems:*
(1) f is continuous on a bounded, closed interval $[a, b]$;
(2) f is differentiable in (a, b);
(3) $f(a) = f(b)$.
Then

- (1)+(2)+(3) $\Longrightarrow$ Rolle's Theorem [(1),(2) and (3) are sufficient conditions for Rolle's Theorem]
- (1)+(2) $\Longrightarrow$ MVT [(1) and (2) are sufficient conditions for MVT]

On the other hand, all conditions (1)-(3) (resp. (1) and (2)) are needed in the Rolle Theorem (resp. MVT). The following are examples of functions you should keep in your mind: they have values if you want to produce counterexamples.

1. $f(x) = \frac{1}{x}$ on the interval $(0, 1]$. f is differentiable, but MVT doesn't apply. [Why?]
2. $f(x) = |x|$ on $[-1, 1]$, f is continuous, differentiable other then the point 0, both MVT and Rolle's do not apply. [Why?]
3. $f(x) = 1$ on $[0, 1]$ and $f(x) = 2$ on $[1, 2]$. f is continuous and differentiable other then at 1. MVT doesn't apply. [Why?]

Theorem 2.27 ***(Cauchy's Mean Value Theorem)*** *Suppose f, $g : [a, b] \to \mathbb{R}$ are continuous, f, g are differentiable on (a, b), and $g' \neq 0$ on (a, b), then there is a point $x_0 \in (a, b)$ such that*

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)} .$$

Proof. The Rolle Theorem implies that $g(b) \neq g(a)$ [Why?]. Apply Rolle's Theorem to function

$$F(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{g(b) - g(a)} (g(x) - g(a)) \right] .$$

Then F is continuous on $[a, b]$, and differentiable in (a, b). Moreover

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(x)$$

and $F(a) = F(b) = 0$. According to Rolle's Theorem, there is a point $\xi \in (a, b)$ such that $F'(\xi) = 0$, that is

$$f'(\xi) = \frac{f(b) - f(a)}{g(b) - g(a)} g'(\xi) \ .$$

Since $g'(\xi) \neq 0$, so that, by dividing $g'(\xi)$ both sides,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)} \ .$$

■

Proposition 2.28 *1)* $\exp(x + y) = \exp(x)\exp(y)$ *for all* $x, y \in \mathbb{R}$.
2) $\exp(x) > 0$ *for any* $x \in (-\infty, +\infty)$, *and therefore* $x \to \exp(x)$ *is strictly increasing,* $\exp(x) \to \infty$ *as* $x \to +\infty$ *and* $\exp(x) \to 0$ *as* $x \to -\infty$. *Therefore the inverse function of* $\exp$ *exists, called the logarithm function, denoted by* $\log$.
3) $\log : (0, \infty) \to (-\infty, \infty)$ *is differentiable, and* $\frac{d}{dx} \log x = \frac{1}{x}$.

Proof. 1) For any (fixed real) c, consider $g(x) = \exp(x)\exp(c - x)$ [Why? Explained in the lecture]. Then

$$\begin{aligned} g'(x) &= \exp'(x)\exp(c - x) - \exp(x)\exp'(c - x) \\ &= \exp(x)\exp(c - x) - \exp(x)\exp(c - x) \\ &= 0 \end{aligned}$$

so that g is constant [Identity Theorem]. Clearly $\exp(0) = 1$, so that $g(x) = g(0) = \exp(c)$ for every x. That is

$$\exp(x)\exp(c - x) = \exp(c) \qquad \forall x \ .$$

Setting $x = a$ and $c = a + b$ we obtain

$$\exp(a + b) = \exp(a)\exp(b) \ .$$

2) If $x \geq 0$ then

$$\begin{aligned} \exp(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \\ &\geq 1 \end{aligned}$$

and if $x < 0$, then

$$1 = \exp(x - x) = \exp(-x)\exp(x)$$

so that

$$0 < \exp(x) = \frac{1}{\exp(-x)} \leq 1 \qquad \forall x < 0 \ .$$

In particular, $\exp(x)$ is strictly increasing on $(-\infty, +\infty)$ as $\exp'(x) = \exp(x) > 0$ [apply MVT to exp on $[x, y]$: see the proof of Corollary 2.25]. Since $\lim_{x\to+\infty} \exp(x) = +\infty$, and $\exp(x) \to 0$ as $x \to -\infty$, the function $f(x) = \exp(x)$ maps $(-\infty, +\infty)$ to $(0, +\infty)$. Thus f has a continuous inverse f^{-1} on $(0, +\infty)$ which is differentiable on $(0, +\infty)$ as $f'(x) = \exp(x) \neq 0$ [Theorem 2.9]. We use $\log(x)$ to denote $f^{-1}(x)$. Suppose $y = f(x) = \exp(x)$, then $f' = f$ so that

$$\log'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f(f^{-1}(y))} = \frac{1}{y}\,.$$

That is, $\frac{d}{dx}\log(x) = \frac{1}{x}$ for any $x > 0$. ■

Exercise 2.29 *Define $e = \exp(1)$. Show that (i) $1 < e < 3$; (ii) e is irrational. [By Contradiction: if $e = \frac{p}{q}$ were rational, where p and q are co-prime, then*

$$\frac{p}{q} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots$$

and you may deduce a contradiction.]

Proposition 2.30 *For $x > 0$ and $a \in \mathbb{R}$, define $x^a = \exp(a \log x)$. Then (i) $x^0 = 1$; (ii) $x^1 = x$; (iii) $x^{a+b} = x^a x^b$ (iv) $x^a y^a = (xy)^{ab}$; (v) $(x^a)^b = x^{ab}$; (vi) $\frac{d}{dx}x^a = ax^{a-1}$. [If n is positive integer, then x^n coincides with the product $x \cdots x$ (n times) as you expect].*

2.2.3 Worked Examples

Example 2.31 *Show that the general solution for $f'(x) = f(x)$; $x \in (0, +\infty)$, is $f(x) = A\exp(x)$ where A is a constant.*

Consider $g(x) = \frac{f(x)}{\exp(x)}$ [Why? Explained in the lecture]. Then

$$\begin{aligned} g'(x) &= \frac{f'(x)\exp(x) - f(x)\exp'(x)}{\exp(x)^2} \\ &= \frac{f(x)\exp(x) - f(x)\exp(x)}{\exp(x)^2} \quad \text{[Use the facts: } \exp' = \exp \text{ and } f' = f\text{]} \\ &= 0 \end{aligned}$$

so that $g = A$ on $(0, +\infty)$ for some constant [Identity Theorem]. Therefore $f(x) = A\exp(x)$ for all $x \in (0, +\infty)$.

Exercise 2.32 *Show that $\sin^2 x + \cos^2 x = 1$ for all $x \in \mathbb{R}$.*

Let $f(x) = \sin^2 x + \cos^2 x$. Then

$$\begin{aligned} f'(x) &= 2\sin x \sin' x + 2\cos x \cos' x \\ &= 2\sin x \cos x + 2\cos x(-\sin x) \\ &= 0 \end{aligned}$$

so that $f(x) = f(0) = 1$ for all $x \in \mathbb{R}$ [Identity Theorem].

Example 2.33 *Show that*

$$\frac{t}{1+t} < \log(1+t) < t \qquad \forall t > 0 \ .$$

By MVT

$$\begin{aligned} \log(1+t) - \log(1) &= \log'(\xi)(1+t-1) \\ &= \frac{t}{\xi} \end{aligned}$$

for some $\xi \in (1, 1+t)$. Thus $1 < \xi < 1+t$, which yields that $\frac{t}{1+t} < \frac{t}{\xi} < t$. Therefore

$$\log(1+t) = \log(1+t) - \log 1 = \frac{t}{\xi}$$

belongs to $(\frac{t}{1+t}, t)$.

Example 2.34 *Show that*

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right)$$

exists. The limit is called Euler's constant, which is $0.57721566490\cdots$.

In Analysis 1, it is showed that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty \qquad \text{(diverges to } \infty) \ .$$

Let

$$\begin{aligned} a_n &= \sum_{k=1}^{n-1} \left(\frac{1}{k} - (\log(k+1) - \log k) \right) \\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \\ &\quad - (\log 2 - \log 1) - (\log 3 - \log 2) - \cdots - (\log n - \log(n-1)) \\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \log n \ . \end{aligned}$$

Apply MVT to $\log x$ and interval $[k, k+1]$: there is a number ξ_k between k and $k+1$ [i.e. $0 < \xi_k - k < 1$] s.t.

$$\log(k+1) - \log k = \frac{1}{\xi_k}(k+1-k) = \frac{1}{\xi_k} \quad .$$

Hence

$$a_n = \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{\xi_k} \right) = \sum_{k=1}^{n-1} \frac{\xi_k - k}{k\xi_k}$$

so that $a_n > 0$ and a_n is increasing. Moreover, since $\xi_k > k$, we thus have

$$a_n < \sum_{k=1}^{n-1} \frac{1}{k^2} < \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty \qquad [\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges for any } p > 1].$$

Hence $\{a_n\}$ is bounded and increasing sequence, so that $\lim_{n\to\infty} a_n = \gamma$ exists [and $\gamma \le \sum_{k=1}^{\infty} \frac{1}{k^2} = \pi^2/6$]. Therefore

$$\begin{aligned} \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) &= \lim_{n\to\infty} \left(a_n + \frac{1}{n}\right) \\ &= \gamma\ . \end{aligned}$$

Example 2.35 *For $x \ge 0$, we have (i)* $\exp(-x) \le 1$; *(ii)* $\exp(-x) \ge 1-x$; *(iii)* $\exp(-x) \le 1 - x + \frac{x^2}{2}$. *In general we have, for any natural number n,*

$$\exp(-x) \le \sum_{k=0}^{2n} (-1)^k \frac{x^k}{k!} \qquad \text{and} \qquad \exp(-x) \ge \sum_{k=0}^{2n+1} (-1)^k \frac{x^k}{k!} \tag{2.7}$$

for any $x \ge 0$.

(i) Let $f(x) = \exp(-x)$. Then $f'(x) = -\exp(-x) < 0$, so that f is decreasing in $[0, +\infty)$. In particular $f(x) \le f(0) = 1$ for all $x \ge 0$.
(ii) Let $g(x) = \exp(-x) - 1 + x$. Then $g'(x) = -\exp(-x) + 1 \ge 0$ [By (i)], so that g is increasing, thus $g(x) \ge g(0) = 0$.
(iii) Consider $h(x) = \exp(-x) - 1 + x - \frac{x^2}{2}$. Then

$$h'(x) = -\exp(-x) + 1 - x \le 0$$

so that h is decreasing in $[0, +\infty)$. Hence $h(x) \le h(0) = 0$.
To prove (2.7) we use induction on n. We have proven the case that $n = 0$. Suppose (2.7) is true for n. Consider $f(x) = \exp(-x) - \sum_{k=0}^{2(n+1)} (-1)^k \frac{x^k}{k!}$. Then

$$\begin{aligned} f'(x) &= \sum_{k=2(n+1)+1}^{\infty} (-1)^k k \frac{x^{k-1}}{k!} = \sum_{k=2(n+1)+1}^{\infty} (-1)^k \frac{x^{k-1}}{(k-1)!} \\ &= -\sum_{k=2(n+1)+1}^{\infty} (-1)^{k-1} \frac{x^{k-1}}{(k-1)!} = -\sum_{k=2(n+1)}^{\infty} (-1)^k \frac{x^k}{(k-1)!} \\ &= -\left(\exp(-x) - \sum_{k=0}^{2n+1} (-1)^k \frac{x^k}{(k-1)!}\right) \\ &\le 0 \qquad\qquad \text{[Induction Assumption]} \end{aligned}$$

so that $f(x)$ is decreasing in $[0, +\infty)$. Hence $f(x) \le f(0) = 0$, that is

$$\exp(-x) \le \sum_{k=0}^{2(n+1)} (-1)^k \frac{x^k}{k!}\ .$$

Similarly $\exp(-x) \ge \sum_{k=0}^{2(n+1)+1} (-1)^k \frac{x^k}{k!}$.

Example 2.36 *Let* $0 < x < \frac{\pi}{2}$. *Then*
1) $\sin x < x < \tan x$ *; [which yields that* $\cos x < \frac{\sin x}{x} < 1$, *so that* $\lim_{x\to 0} \frac{\sin x}{x} = 1$.*]*
2) $\frac{2}{\pi} < \frac{\sin x}{x} < 1$. *[1) + 2) implies that* $\max\{\cos x, \frac{2}{\pi}\} < \frac{\sin x}{x} < 1$ *for* $x \in (0, \pi/2)$*].*

To prove the first inequality, consider $f(x) = \tan x - x$, $x \in [0, \pi/2)$. Then f is differentiable on $(0, \pi/2)$ and

$$f'(x) = \frac{1}{\cos^2 x} - 1 > 0 \qquad \forall x \in (0, \pi/2) .$$

f is strictly increasing [Apply MVT to any $[x_1, x_2]$, where $x_i \in (0, \pi/2)$]. Thus $f(x) > f(0)$ for any $x \in (0, \pi/2)$ which yields the inequality 1).
2) If $g(x) = x - \sin x$ then $g'(x) = 1 - \cos x > 0$ for any $x \in (0, \pi/2)$. Hence g is strictly increasing on $[0, \pi/2]$, so that $\sin x < x$ for all $x \in (0, \pi/2)$. Now consider

$$h(x) = \frac{\sin x}{x} \qquad x \in (0, \pi/2] .$$

Then

$$h'(x) = \frac{\cos x(x - \tan x)}{x^2} < 0 \qquad \forall x \in (0, \pi/2)$$

so that h is strictly decreasing, so that $g(x) > g(\pi/2)$ for any $x \in (0, \pi/2)$.

Example 2.37 *(i) Suppose* f *is continuous in* $[x_0, x_0 + \delta]$ *and differentiable in* $(x_0, x_0 + \delta)$ *for some* $\delta > 0$ *and suppose* $\lim_{x\to x_0+} f'(x)$ *exists, then the right-derivative of* f *at* x_0 *exists and*

$$f'(x_0+) = \lim_{x\to x_0+} f'(x) .$$

[Remind you that, here, $f'(x_0+)$ *does not mean the right-hand limit of the derivative function* f', *but the limit*

$$\lim_{x\downarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} .$$

It shows that, if the right-hand limit of f' *exists, i.e.* $\lim_{x\downarrow x_0} f'(x)$ *exists, then* $\lim_{x\downarrow x_0} f'(x)$ *coincides with* $f'(x_0+)$, *which justify the abuse of notations]. In particular, if* $\lim_{x\to x_0} f'(x)$ *exists, then* f *is differentiable at* x_0, *and* $f'(x_0) = \lim_{x\to x_0} f'(x)$ *[However,* f *can be differentiable at* x_0, *but* $\lim_{x\to x_0} f'(x)$ *doesn't exist. Example?]*
(ii) Show that $f(x) = x \arcsin x + \sqrt{1 - x^2}$ *is differentiable on* $[-1, 1]$. *[*$\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ *is the inverse of* $\sin$, *and* $\sqrt{x}$ *is the inverse of* x^2 *in* $[0, +\infty)$*].*

(i) Indeed, for any $x \in (x_0, x_0 + \delta)$ we apply the MVT to f on $[x_0, x]$

$$f(x) - f(x_0) = f'(\xi_x)(x - x_0) .$$

Clearly, as $x \to x_0$, $\xi_x \to x_0$ so that $\lim_{x\downarrow x_0} f'(\xi_x) = \lim_{x\downarrow x_0} f'(x)$, and therefore

$$\begin{aligned} f'(x_0+) &= \lim_{x\downarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x\downarrow x_0} f'(\xi_x) = \lim_{x\downarrow x_0} f'(x) . \end{aligned}$$

(ii) First let us compute the derivative of arcsin on $(-1,1)$. According to Theorem 2.9

$$\begin{aligned}\frac{d}{dx}\arcsin x &= \frac{1}{\sin'(\arcsin x)}\\ &= \frac{1}{\cos(\arcsin x)}\ .\end{aligned}$$

Since sin is increasing in $[-\frac{\pi}{2},\frac{\pi}{2}]$, so its inverse arcsin is continuous on $[-1,1]$ with values in $[-\frac{\pi}{2},\frac{\pi}{2}]$. In particular $\cos(\arcsin x)\geq 0$. Since $\cos^2+\sin^2=1$, so that

$$\begin{aligned}\cos(\arcsin x) &= \sqrt{1-(\sin(\arcsin x))^2}\\ &= \sqrt{1-x^2}\ .\end{aligned}$$

Therefore [Theorem 2.9]

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \forall x\in(-1,1)\ .$$

[Exercise: Carefully work out the derivative $\frac{d}{dx}\sqrt{x}$ via Theorem 2.9]. Hence

$$f'(x)=\arcsin x+\frac{x}{\sqrt{1-x^2}}-\frac{x}{\sqrt{1-x^2}}=\arcsin x$$

on $(-1,1)$. However $\lim_{x\to\pm1} f'(x)=\pm\frac{\pi}{2}$ exist, so that $f'(-1+)=-\frac{\pi}{2}$ and $f'(1-)=\frac{\pi}{2}$. f is differentiable in $[-1,1]$.

Theorem 2.38 ***(Darboux' Intermediate Value Theorem)*** *If $f:[a,b]\to\mathbb{R}$ is differentiable and $f'(a)<A<f'(b)$, then there exists a point $\xi\in(a,b)$ such that $f'(\xi)=A$.*

Proof. Let $g(x)=f(x)-Ax$. Then g is differentiable in $[a,b]$, so that g is continuous in $[a,b]$. Therefore g attains its bounds. Moreover

$$g'(x)=f'(x)-A$$

so that $g'(a)=f'(a)-A<0$ and $g'(b)=f'(b)-A>0$. Since $g'(a)<0$ there exists $x_1\in(a,b)$ such that $g(x_1)<g(a)$ [Why? (Exercise, explained in the lecture)]. Similarly, as $g'(b)>0$, there is $x_2\in(a,b)$ such that $g(x_2)<g(b)$. Therefore g must have its minimum $\xi\in(a,b)$. By Rolle's Theorem, $g'(\xi)=0$. ∎

Example 2.39 *Consider $f(x)=x^2\sin\frac{1}{x}$ if $x\neq 0$, and $f(0)=0$. f is differentiable everywhere, but the derivative function*

$$f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$$

is not continuous at 0, and thus IVT [Chapter 1: IVT for continuous functions on closed intervals] does not apply to f' on $[-1,1]$ for example, but f' attains every values between $f'(-1)$ and $f'(1)$ by Darboux IVT.

2.3 L'Hôpital rule

[Theorems of G. F. de l'Hospitales, French mathematician, and Joh. Bernoulli] In this section, all functions are real-valued functions.

Theorem 2.40 *Suppose f, g are continuous on $[a, a+\delta]$ (for some $\delta > 0$), differentiable in $(a, a+\delta)$, and $f(a) = g(a) = 0$, then*

$$\lim_{x \downarrow a} \frac{f(x)}{g(x)} = \lim_{x \downarrow a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right-hand side exists.

Proof. Let

$$l = \lim_{x \downarrow a} \frac{f'(x)}{g'(x)} .$$

For every $x \in (a, a+\delta)$, apply Cauchy's MVT to f, g on the interval $[a, x]$: there is $\xi_x \in (a, x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi_x)}{g'(\xi_x)}$$

Letting $x \downarrow a$, then $\xi_x \to a$ and $\xi_x > a$, so that

$$\lim_{x \downarrow a} \frac{f'(\xi_x)}{g'(\xi_x)} = l,$$

hence

$$\lim_{x \downarrow a} \frac{f(x)}{g(x)} = \lim_{x \downarrow a} \frac{f'(\xi_x)}{g'(\xi_x)} = l \qquad \text{[Algebra of limits].}$$

■

Similarly

Theorem 2.41 *Suppose f, g are continuous on $[a-\delta, a]$ (for some $\delta > 0$) and differentiable on $(a-\delta, a)$, $f(a) = g(a) = 0$, then*

$$\lim_{x \uparrow a} \frac{f(x)}{g(x)} = \lim_{x \uparrow a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right-hand side exists.

Theorem 2.42 ***(L'Hôpital Rule)*** *Suppose f, g are continuous on $[a-\delta, a+\delta]$ (for some $\delta > 0$) and differentiable on $(a-\delta, a+\delta) \setminus \{a\}$, $f(a) = g(a) = 0$, then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right-hand side exists.

Example 2.43 *Show that (i)* $\lim_{x\to 0} \frac{\sin x}{x} = 1$; *(ii)* $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$; *(iii)* $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$; *(iv)* $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$; *(v) Find* $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$.

(i) ($\frac{0}{0}$ type) $\sin x$ and x are continuous, with values 0 at 0. Since

$$\lim_{x\to 0}\frac{\sin' x}{x'} = \lim_{x\to 0}\cos x = 1$$

exists, so that

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\sin' x}{x'} = 1 \qquad\qquad \text{[L'Hôpital Rule]}.$$

(ii) ($\frac{0}{0}$ type) We have

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\frac{\sin x}{2x} \quad \text{[provided this limit exists]}\\ &= \lim_{x\to 0}\frac{\cos x}{2} \quad \text{[provided this limit exists]}\\ &= \frac{1}{2}\ .\end{aligned}$$

We have used L'Hôpital Rule twice.
(iii) ($\frac{0}{0}$ type) By L'Hôopital Rule: $\log(1+x)$ is continuous at 0 and $\log(1+0)=0$.

$$\begin{aligned}\lim_{x\to 0}\frac{\log(1+x)}{x} &= \lim_{x\to 0}\frac{\log'(1+x)}{x'} \quad \text{[provided this limit exists]}\\ &= \lim_{x\to 0}\frac{1}{1+x} = 1\ .\end{aligned}$$

(iv) (1^∞ type $\Longrightarrow \exp(\frac{0}{0})$ type, then use the continuity of exp) According the definition a^p,

$$(1+x)^{\frac{1}{x}} = \exp\left(\frac{1}{x}\log(1+x)\right)$$

Since exp is continuous at 1, so that [By (iii)]

$$\begin{aligned}\lim_{x\to 0}(1+x)^{\frac{1}{x}} &= \lim_{x\to 0}\exp\left(\frac{\log(1+x)}{x}\right)\\ &= \exp\left(\lim_{x\to 0}\frac{\log(1+x)}{x}\right) \qquad [\exp \text{ is continuous at } 1]\\ &= \exp(1) = e\ .\end{aligned}$$

Example 2.44 $\lim_{x\to 0}(1+ax)^{\frac{1}{x}} = \exp(a)$ *for any* $a\in\mathbb{R}$. *In particular*

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = \exp(a)\ .$$

If $a=0$, then $\lim_{x\to 0}(1+ax)^{\frac{1}{x}} = \lim_{x\to 0}1 = 1 = \exp(0)$. If $a\neq 0$, then

$$\begin{aligned}\lim_{x\to 0}(1+ax)^{\frac{1}{x}} &= \lim_{x\to 0}\exp\left(\frac{1}{x}\log(1+ax)\right) \qquad \text{[By definition]}\\ &= \exp\left(\lim_{x\to 0}\frac{1}{x}\log(1+ax)\right) \qquad [\text{Continuity of } \exp]\\ &= \exp\left(\lim_{x\to 0}\frac{a}{(1+ax)}\right) \qquad \text{[if the limit exists, L'Hôpital Rule]}\\ &= \exp(a)\ .\end{aligned}$$

Theorem 2.45 *If $f, g : (a, a+\delta) \to \mathbb{R}$ are differentiable, where $\delta > 0$, $g'(x) \neq 0$, $f(x) \to \infty$, $g(x) \to \infty$ as $x \downarrow a$, and $\lim_{x\downarrow a} \frac{f'(x)}{g'(x)}$ exists (or $\pm\infty$), then*

$$\lim_{x\downarrow a} \frac{f(x)}{g(x)} = \lim_{x\downarrow a} \frac{f'(x)}{g'(x)} \ .$$

Proof. Suppose that $\lim_{x\downarrow a} \frac{f'(x)}{g'(x)} = K$ is finite [Otherwise we may consider $\lim_{x\downarrow a} \frac{g(x)}{f(x)}$ instead]. We may assume that $g' \neq 0$ [That $g' \neq 0$ near a is implied in the assumption that $\lim_{x\downarrow a} \frac{f'(x)}{g'(x)}$ exists]. $\forall \varepsilon > 0$ there is a number δ_1 ($< \delta$) such that

$$\left| \frac{f'(x)}{g'(x)} - K \right| < \frac{\varepsilon}{2} \qquad \forall x \in (a, a+\delta_1) \ . \tag{2.8}$$

Now we choose a number c in $(a, a+\delta_1)$ [c is fixed from now on]. For any $x \in (a, c)$ we apply Cauchy's MVT to f, g on $[x, c]$: there is a number $\xi_x \in (x, c)$ such that

$$\frac{f(c) - f(x)}{g(c) - g(x)} = \frac{f'(\xi_x)}{g'(\xi_x)} \ .$$

Since $\xi_x \in (x, c) \subset (a, a+\delta_1)$, by (2.8)

$$\left| \frac{f(x) - f(c)}{g(x) - g(c)} - K \right| = \left| \frac{f'(\xi_x)}{g'(\xi_x)} - K \right| < \frac{\varepsilon}{2} \qquad \forall x \in (a, c) \ . \tag{2.9}$$

[However, we cannot conclude from (2.9) that $\frac{f(x)-f(c)}{g(x)-g(c)} \to K$ as $x \downarrow a$ (although it does !!), as there is no guarantee that ξ_x will tend to a as $x \downarrow a$]. Now we consider

$$\begin{aligned}
\frac{f(x)}{g(x)} - K &= \frac{f(x) - f(c) + f(c)}{g(x)} - K \\
&= \frac{f(c)}{g(x)} + \frac{f(x) - f(c)}{g(x) - g(c)} \frac{g(x) - g(c)}{g(x)} - K \\
&= \frac{f(c)}{g(x)} + \frac{f(x) - f(c)}{g(x) - g(c)} \left(1 - \frac{g(c)}{g(x)}\right) - K \\
&= \frac{f(c)}{g(x)} + \left(\frac{f(x) - f(c)}{g(x) - g(c)} - K\right) \left(1 - \frac{g(c)}{g(x)}\right) + K \left(1 - \frac{g(c)}{g(x)}\right) - K \\
&= \frac{f(c)}{g(x)} + \left(\frac{f(x) - f(c)}{g(x) - g(c)} - K\right) \left(1 - \frac{g(c)}{g(x)}\right) - \frac{Kg(c)}{g(x)} \\
&= \frac{f(c) - Kg(c)}{g(x)} + \left(1 - \frac{g(c)}{g(x)}\right) \left(\frac{f(x) - f(c)}{g(x) - g(c)} - K\right)
\end{aligned}$$

[Why we are interested in this? Explained in the lecture], so that

$$\begin{aligned}
\left| \frac{f(x)}{g(x)} - K \right| &\leq \left| \frac{f(c) - Kg(c)}{g(x)} \right| + \left| 1 - \frac{g(c)}{g(x)} \right| \left| \frac{f(x) - f(c)}{g(x) - g(c)} - K \right| \\
&\leq \left| \frac{f(c) - Kg(c)}{g(x)} \right| + \frac{\varepsilon}{2} \left| 1 - \frac{g(c)}{g(x)} \right|
\end{aligned}$$

for any $x \in (a, c)$. Since $g(x) \to \infty$ as $x \downarrow a$ so that

$$\lim_{x \downarrow a} \frac{f(c) - Kg(c)}{g(x)} = 0$$

and

$$\lim_{x \downarrow a} \left(1 - \frac{g(c)}{g(x)}\right) = 1 \ .$$

[Algebra of limits]. Thus there is $\delta_2 > 0$ [and $\delta_1 < \min\{\delta_1, c - a\}$] such that

$$\left|1 - \frac{g(c)}{g(x)}\right| < \frac{4}{3} \qquad \text{and} \qquad \left|\frac{f(c) - Kg(c)}{g(x)}\right| < \frac{\varepsilon}{3}$$

for every $x \in (a, a + \delta_1)$. Therefore

$$\left|\frac{f(x)}{g(x)} - K\right| < \frac{\varepsilon}{3} + \frac{4}{3}\frac{\varepsilon}{2} = \varepsilon \qquad \forall x \in (a, a + \delta_1) \ .$$

By definition, $\lim_{x \downarrow a} \frac{f(x)}{g(x)} = K$. ∎

Theorem 2.46 *Suppose $f, g : (a, +\infty) \to \mathbb{R}$ are continuous and differentiable, with $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$. If $g'(x) \neq 0$ on $(a, +\infty)$ and $\frac{f'(x)}{g'(x)} \to l$, then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = l$.*

Proof. Apply **L'Hôpital Rule** to functions $F(x) = f(\frac{1}{x})$ and $G(x) = g(\frac{1}{x})$. ∎

Example 2.47 $\lim_{x \to +\infty} \frac{\log x}{x^\mu} = 0$ *[$\frac{\infty}{\infty}$ type] and* $\lim_{x \to +\infty} \frac{x^\mu}{e^x} = 0$ *[$\frac{\infty}{\infty}$ type] for any $\mu > 0$.*

Let $g(x) = x^\mu = \exp(\mu \log x)$. Then $g'(x) = \mu x^{\mu - 1}$. By L'Hôpital rule

$$\begin{aligned} \lim_{x \to +\infty} \frac{\log x}{x^\mu} &= \lim_{x \to +\infty} \frac{\frac{1}{x}}{\mu x^{\mu - 1}} \quad \text{[provided this limit exists]} \\ &= \lim_{x \to +\infty} \frac{1}{\mu x^\mu} = 0 \ . \end{aligned}$$

Example 2.48 *For any $\mu > 0$,* $\lim_{x \downarrow 0} x^\mu \log x = 0$ *. [$0 \cdot \infty$ type $\Longrightarrow \frac{\infty}{\infty}$ type]*

Again use L'Hôpital Rule

$$\begin{aligned} \lim_{x \downarrow 0} x^\mu \log x &= \lim_{x \downarrow 0} \frac{\log x}{x^{-\mu}} \\ &= \lim_{x \downarrow 0} \frac{\log' x}{(x^{-\mu})'} \qquad \text{[if this limit exists]} \\ &= \lim_{x \downarrow 0} \frac{\frac{1}{x}}{(-\mu) x^{-\mu - 1}} = \lim_{x \downarrow 0} \frac{x^\mu}{(-\mu)} = 0 \ . \end{aligned}$$

Example 2.49 *Show that*

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}} = \frac{1}{\sqrt[3]{e}} \ .$$

[1^∞ type $\Longrightarrow$ exp $\left(\frac{0}{0}\text{ type}\right)$, then use the continuity of exp] Since $f(x) = \left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}$ is even function, so that we only need to show that $\lim_{x\downarrow 0} f(x) = \frac{1}{\sqrt[3]{e}}$. According to definition

$$\begin{aligned} f(x) &= \exp\left(\frac{1}{1-\cos x}\log\frac{\sin x}{x}\right) \\ &= \exp\left(\frac{\log\sin x - \log x}{1-\cos x}\right) \ . \end{aligned}$$

By L'Hôpital Rule,

$$\begin{aligned} \lim_{x\downarrow 0}\frac{\log\sin x - \log x}{1-\cos x} &= \lim_{x\downarrow 0}\frac{\frac{\cos x}{\sin x}-\frac{1}{x}}{\sin x} \qquad \text{[provided it exists]} \\ &= \lim_{x\downarrow 0}\frac{x\cos x - \sin x}{x\sin^2 x} \\ &= \lim_{x\downarrow 0}\frac{\cos x - x\sin x - \cos x}{\sin^2 x + 2x\sin x\cos x} \qquad \text{[if exists, use L'Hôpital again]} \\ &= -\lim_{x\downarrow 0}\frac{x}{\sin x + 2x\cos x} \\ &= -\lim_{x\downarrow 0}\frac{1}{\cos x + 2\cos x - 2x\sin x} \\ &= -\frac{1}{3} \ . \end{aligned}$$

Since exp is continuous at $-\frac{1}{3}$, so that

$$\begin{aligned} \lim_{x\downarrow 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}} &= \lim_{x\downarrow 0}\exp\left(\frac{\log\sin x - \log x}{1-\cos x}\right) \\ &= \exp\left(\lim_{x\downarrow 0}\frac{\log\sin x - \log x}{1-\cos x}\right) \qquad \text{[by continuity of exp]} \\ &= \exp\left(-\frac{1}{3}\right) \ . \end{aligned}$$

2.4 Taylor's formula

If f is a function defined on $[a,b]$ (where $a < b$) which has all (right-hand) derivatives $f^{(k)}(a)$ of orders up to $n-1$ (where $n \geq 1$ is an integer, with convention that $f^{(0)} = f$), then we may form a polynomial of degree $n-1$:

$$P_{n-1}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{(n-1)} \ .$$

Then $P_{n-1}^{(k)}(a) = f^{(k)}(a)$ for all $k \leq n-1$, that is, P_{n-1} agrees with f at a up to $n-1$-th order derivative.

$$\begin{aligned} P_0(x) &= f(a) \qquad \text{[constant polynomial, not interesting];} \\ P_1(x) &= f(a) + f'(a)(x-a) \qquad \text{[linear approximation to } f \text{ near } a\text{];} \\ P_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \quad \text{[quadratic approximation about } a\text{];} \\ &\cdots . \end{aligned}$$

Let

$$\begin{aligned} E_n(x,a) &= f(x) - P_{n-1}(x) \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned} \tag{2.10}$$

be the error between $f(x)$ and $P_{n-1}(x)$. If in addition, f has derivatives $f^{(n)}$ at a for all n, then we may form a power series

$$\begin{aligned} P(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned} \tag{2.11}$$

which is called the Taylor expansion of f at a. The following lemma is obvious.

Lemma 2.50 *Let $f : [a,b] \to \mathbb{R}$ be differentiable up to any degree [$f^{(n)}(a)$ exists for any n], R be the convergence radius of the Taylor expansion (2.11), and $x \in [a,b]$. Then*

$$f(x) = P(x)$$

if and only if $E_n(a,x) \to 0$ as $n \to \infty$. In this case, we must have $|x-a| \leq R$.

It is therefore quite important to derive a useful formula for the error $E_n(a,x)$ which is achieved in the following Taylor's theorem.

Theorem 2.51 ***(Taylor's Theorem)*** *Let $f : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, where $b > a$. Suppose $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists on (a,b). Then there is a number $\xi \in (a,b)$ such that*

$$\begin{aligned} f(b) &= P_{n-1}(b) + \frac{f^{(n)}(\xi)}{n!}(b-a)^n \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n)}(\xi)}{n!}(b-a)^n . \end{aligned}$$

That is,

$$E_n(a,b) = \frac{f^{(n)}(\xi)}{n!}(b-a)^n$$

for some ξ [depending on a, b and n as well] between a and b, which is called the remainder in Lagrange form.

[In general, ξ may depend on a, b and n as well].

Proof. We use the method of "varying a constant": regard a in the definition of $P_{n-1}(x)$ as a variable. We therefore consider the function

$$\begin{aligned} F(x) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(b-x)^k \\ &= f(x) + f'(x)(b-x) + \frac{f''(x)}{2!}(b-x)^2 + \cdots + \frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{n-1} \end{aligned}$$

for $x \in [a,b]$. Then $F(b) = f(b)$ and $F(a) = P_{n-1}(b)$. Let $G(x) = (b-x)^n$. Then F and G are continuous on $[a,b]$, differentiable on (a,b). Moreover

$$\begin{aligned} F'(x) &= \sum_{k=0}^{n-1} \frac{f^{(k+1)}(x)}{k!}(b-x)^k + \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!}(-1)\,k\,(b-x)^{k-1} \qquad \text{[Product Rule]} \\ &= \sum_{k=0}^{n-1} \frac{f^{(k+1)}(x)}{k!}(b-x)^k - \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{(k-1)!}(b-x)^{k-1} \\ &= \frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1} \end{aligned}$$

and

$$G'(x) = -n(b-x)^{n-1} .$$

Hence $G' \neq 0$ on (a,b). Apply Cauchy's MVT to F, G on $[a,b]$: there is a number $\xi \in (a,b)$ such that

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{\frac{f^{(n)}(\xi)}{(n-1)!}(b-\xi)^{n-1}}{-n(b-\xi)^{n-1}} = -\frac{f^{(n)}(\xi)}{n!} .$$

Substituting the values $F(a)$, $F(b)$, $G(a)$ and $G(b)$ into the above equation we obtain

$$\frac{f(b)-P_{n-1}(b)}{-(b-a)^n} = -\frac{f^{(n)}(\xi)}{n!}$$

which can be rewritten as

$$f(b) = P_{n-1}(b) + \frac{f^{(n)}(\xi)}{n!}(b-a)^n .$$

■

Remark 2.52 *We may use the same argument with* any *function G which is continuous in $[a,b]$, differentiable in (a,b), and $G' \neq 0$. Then, according to Cauchy's MVT, there is a number ξ between a and b, such that*

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{\frac{f^{(n)}(\xi)}{(n-1)!}(b-\xi)^{n-1}}{G'(\xi)}$$

so that

$$f(b) = P_{n-1}(b) + \frac{f^{(n)}(\xi)}{(n-1)!}(b-\xi)^{n-1}\frac{G(b)-G(a)}{G'(\xi)}.$$

By choosing different function G, you may devise Taylor's Theorem with the remainder of different forms. For example, if we choose $G(x) = x - a$, then $\frac{G(b)-G(a)}{G'(\xi)} = b - a$. Thus

$$f(b) = P_{n-1}(b) + \frac{f^{(n)}(\xi)}{(n-1)!}(b-a)(b-\xi)^{n-1}$$

for some $\xi \in (a, b)$. You may for example try $G(x) = (x-a)^m$ for a power $m \geq 1$ to see what kind of Taylor's formula you can get. Of course, if you choose different G, you will have different ξ between a and b.

If we set $b - a = h$, then Taylor's theorem may be stated as

$$f(a+h) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}h^k + \frac{f^{(n)}(a+\theta h)}{n!}h^n$$

where θ is some number between 0 and 1, which depends on a, h and n.
Given a function f which has derivatives of all orders near a, so that you may write down the sequence of $\{f^{(k)}(a)\}$ and the power series [called the Taylor expansion of f at a]

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \qquad (2.12)$$

The power series has convergence radius R, so that the power series (2.12) defines a function on $(a-R, a+R)$ [and in general, you have to use other methods to study the convergence at $a-R$ and $a+R$], denoted by g. That is

$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \forall x \in (a-R, a+R). \qquad (2.13)$$

If it happens $R = 0$, then the Taylor expansion (2.13) is useless for the study of f. Otherwise, all derivatives of the Taylor expansion (2.13) g at a coincide with those of f at a: $g^{(n)}(a) = f^{(n)}(a)$ for any n [Differentiating a power series term by term again and again]. We therefore have a high hope that $f(x) = g(x)$ for all $x \in (a-R, a+R)$. However, this dream can be easily destroyed: the formation of the Taylor expansion (2.13) relies only on the values of f in an arbitrary small neighborhood about a, say $(a-\varepsilon, a+\varepsilon)$ for whatever how small $\varepsilon > 0$, and there is absolutely no reason why we should have $f(x) = g(x)$ if $x \neq a$, UNLESS $f(x)$ CAN BE DETERMINED by the values of f near a [and through the Taylor expansion of course!] This is the concept of analytic functions which will be studied in PART A: Analysis.

Example 2.53 *Let $f(x) = \exp(-\frac{1}{x^2})$ if $x \neq 0$ and $f(0) = 0$. Then f has derivatives of all order, and $f^{(n)}(0) = 0$ for all n. In fact, for $x \neq 0$, we have*

$$f^{(n)}(x) = Q_n(x)\exp(-\frac{1}{x^2})$$

for some polynomial Q_n *of* $\frac{1}{x}$, *so that* $\lim_{x\to 0} f^{(n)}(x) = 0$ *for any* n *[L'Hôpital Rule]. Hence* $f^{(n)}(0) = 0$ *[Example 2.37]. Thus*

$$f(0+h) \neq f(0) + f'(0)h + \cdots + \frac{f^{(n)}(0)}{n!}h^n + \cdots$$

as the right-hand side is identically zero. The remainder $E_n(0,h)$ *for this function does not tend to* 0 *as* $n \to \infty$ *for any* $h \neq 0$. *In this case, we say* f *is not analytic at* 0.

The Taylor's Theorem also provides us with an explicit error estimate between $f(x)$ and its Taylor approximation

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \ .$$

Corollary 2.54 *Let* $f : [a,b] \to \mathbb{R}$ *have continuous derivatives of all orders on* $[a,b]$, *and*

$$E_n = \frac{|b-a|^n}{n!} \sup_{\xi\in[a,b]} |f^{(n)}(\xi)|.$$

Then

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \right| \leq E_n \qquad \forall x \in [a,b] \ .$$

In particular, if $E_n \to 0$ *as* $n \to \infty$ *then*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \qquad \textit{uniformly on } [a,b] \ .$$

Theorem 2.55 *We have*

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} \qquad \forall x \in (-1,1] \ . \tag{2.14}$$

In particular

$$\log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \ .$$

Proof. Consider $f(x) = \log(1+x)$. Then $f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$, so that

$$f(x) = \sum_{k=1}^{n-1} (-1)^{k-1}\frac{x^k}{k} + E_n(x)$$

where, by Taylor's Theorem

$$E_n(x) = \frac{x^n}{n!} f^{(n)}(\xi_n) = (-1)^{n-1}\frac{1}{n}\left(\frac{x}{1+\xi_n}\right)^n$$

for some ξ_n between 0 and x [which depends on x and n as well]. Clearly

$$|E_n(x)| = \frac{1}{n}\left|\frac{x}{1+\xi_n}\right|^n$$

thus, $E_n(x) \to 0$ if $\left|\frac{x}{1+\xi_n}\right| \leq 1$ for all n. The convergence radius of $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$ is 1 [Ratio Test, Analysis I], we must have $|x| \leq 1$ in order that $E_n(x) \to 0$.
Now analyze the condition that $\left|\frac{x}{1+\xi_n}\right| \leq 1$ by keeping in mind that $|x| \leq 1$, $|\xi_n| < 1$ and ξ_n is between 0 and x. The inequality $\left|\frac{x}{1+\xi_n}\right| \leq 1$ is thus equivalent to that

$$|x| \leq 1 + \xi_n$$

that is

$$\xi_n \geq |x| - 1 \ .$$

which is true if $x \in [-\frac{1}{2}, 1]$. Therefore

$$\log(1+x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k} \qquad \text{for } x \in [-\frac{1}{2}, 1] \ . \tag{2.15}$$

[As a byproduct, we thus proved that the power series $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$ is convergent at $x = 1$].
However we fail to prove that $E_n(x) \to 0$ for $x \in (-1, -\frac{1}{2})$ (it does tend to zero !), BECAUSE WE ARE LACK OF ENOUGH INFORMATION ABOUT ξ_n to make a conclusion. In order to handle the values $x \in (-1, -\frac{1}{2})$, let us consider the function

$$g(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \qquad \forall x \in (-1, 1] \ .$$

Then g is differentiable on $(-1, 1)$ and g' can be determined by differentiating the power series term by term [Theorem 2.10]:

$$\begin{aligned} g'(x) &= \sum_{n=1}^{\infty}(-1)^{n-1}n\frac{x^{n-1}}{n} \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1} \\ &= \frac{1}{1-(-x)} = \frac{1}{1+x} \qquad \forall x \in (-1, 1). \end{aligned}$$

On the other hand $f'(x) = \frac{d}{dx}\log(1+x) = \frac{1}{1+x}$ on $(-1, 1)$, and thus $f' = g'$ on $(-1, 1)$. By Identity Theorem

$$f(x) - g(x) = \text{constant} = f(0) - g(0) = 0$$

so that

$$\log(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \qquad \forall x \in (-1, 1) \ .$$

Together with (2.15) we thus have

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad \forall x \in (-1,1] \ .$$

■

Theorem 2.56 *(The Binomial Expansion) Let p be a real number. Then*

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \cdots$$

for any $x \in (-1,1)$.

Proof. Let $f(x) = (1+x)^p$ for $x > -1$. Then

$$\begin{aligned} f'(x) &= p(1+x)^{p-1} \quad ; \\ f''(x) &= p\,(p-1)\,(1+x)^{p-2} \ ; \\ &\cdots \ ; \\ f^{(k)}(x) &= p(p-1)\cdots(p-(k-1))(1+x)^{p-k} \end{aligned}$$

and thus $f^{(k)}(0) = p(p-1)\cdots(p-(k-1))$. By Taylor's Theorem, for any $x > -1$, there is a number ξ_n between 0 and x such that

$$(1+x)^p = 1 + \sum_{k=1}^{n-1} \frac{p(p-1)\cdots(p-(k-1))}{k!}x^k + E_n(x)$$

where

$$\begin{aligned} E_n(x) &= \frac{p(p-1)\cdots(p-(n-1))}{n!}(1+\xi_n)^{p-n}x^n \\ &= \frac{p(p-1)\cdots(p-(n-1))}{n!}(1+\xi_n)^p \left(\frac{x}{1+\xi_n}\right)^n \ . \end{aligned}$$

If $x \geq 0$, then $\xi_n > 0$ so that

$$\begin{aligned} |E_n(x)| &\leq \frac{|p(p-1)\cdots(p-(n-1))|}{n!}|1+x|^p x^n \\ &\equiv a_n \ . \end{aligned}$$

Since

$$\frac{a_{n+1}}{a_n} = \frac{|p-n|}{(n+1)}x \to x \qquad \text{as } n \to \infty \ .$$

By the ratio test, if $|x| < 1$ then the series $\sum a_n$ converges, so that $a_n \to 0$. Hence $E_n(x) \to 0$ if $0 \leq x < 1$. Therefore

$$(1+x)^p = 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!}x^n \qquad \forall x \in [0,1) \ .$$

To extended to $(-1,0)$ we consider

$$g(x)=1+\sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{n!}x^n$$

which is a power series with convergence radius $R=1$, so that g is differentiable on $(-1,1)$ and

$$g'(x)=\sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{(n-1)!}x^{n-1}\ .$$

Hence

$$\begin{aligned}(1+x)\,g'(x) &= \sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{(n-1)!}(1+x)\,x^{n-1}\\ &= \sum_{n=0}^{\infty}\frac{p(p-1)\cdots(p-n)}{n!}x^n+\sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{n!}nx^n\\ &= p+\sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{n!}((p-n)+n)\,x^n\\ &= p+p\sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{n!}x^n\\ &= pg(x)\ .\end{aligned}$$

It is clear that $(1+x)f'(x)=pf(x)$ and $f>0$ for all $x>-1$. Let $h=g/f$ on $(-1,1)$. Then

$$\begin{aligned}h' &= \frac{f'g-g'f}{f^2}\\ &= \frac{(1+x)f'g-(1+x)g'f}{(1+x)f^2}\\ &= \frac{pfg-pgf}{(1+x)f^2}=0\end{aligned}$$

so that g/f is constant in $(-1,1)$, and therefore [The Identity Theorem]

$$\frac{g(x)}{f(x)}=\frac{g(0)}{f(0)}=1 \qquad \forall x\in(-1,1)\ .$$

Hence

$$(1+x)^p=1+\sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{n!}x^n \qquad x\in(-1,1)\ .$$

■

The cases $x=1$ or $x=-1$ need to be considered separately. The conclusion are the following. Their proofs will not presented in my lectures due to time constrain.

Theorem 2.57 *1) If $p > 0$ [Note that, $(1+x)^p$ is continuous at 1 and -1], then*

$$(1+x)^p = 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n \tag{2.16}$$

for all $x \in [-1,1]$. For example

$$\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-(n-1))}{n!} x^n \qquad \forall x \in [-1,1] \ .$$

2) If $-1 < p < 0$ [Note that, in this case, $(1+x)^p$ is not continuous at -1], (2.16) is true for $x \in (-1,1]$. In particular, if $p \in (-1,0)$ then

$$\sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} \tag{2.17}$$

converges. In particular

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-(n-1))}{n!} x^n \qquad \forall x \in (-1,1]$$

3) If $p \leq -1$ then (2.17) is divergent, so that the expansion (2.16) is only true for $x \in (-1,1)$. For example

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad \textit{only for } x \in (-1,1)$$

not true at -1 or 1.

Proof. 3) was proved in the previous theorem. According to the Taylor Theorem

$$(1+x)^p = 1 + \sum_{n=1}^{n-1} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n + E_n(x)$$

where

$$\begin{aligned} E_n(x) &= \frac{p(p-1)\cdots(p-(n-1))}{n!}(1+\xi_n)^{p-n} x^n \\ &= \frac{p(p-1)\cdots(p-(n-1))}{n!}(1+\xi_n)^p \left(\frac{x}{1+\xi_n}\right)^n \end{aligned}$$

for some ξ_n between 0 and x (note that ξ depends on x and n).
Step 1. [Question 4 in Problem 8 is a special case of this part] If $p \in (0,1)$, then we may rewrite the remainder

$$E_n(x) = (-1)^{n-1} \frac{p}{n} \frac{1-p}{1} \frac{2-p}{2} \cdots \frac{(n-1)-p}{n-1} (1+\xi_n)^p \left(\frac{x}{1+\xi_n}\right)^n \ .$$

If $x \in [0,1]$, then $\xi_n \in (0,1)$ so that

$$0 \le \frac{x}{1+\xi_n} < 1, \quad (1+\xi_n)^p \le 2^p \ .$$

Since, when $p \in (0,1)$, each term $\frac{k-p}{k} \le 1$, we thus have

$$|E_n(x)| \le \frac{p}{n} 2^p \to 0 \qquad \text{as } n \to \infty$$

for any $p \in (0,1)$ and $x \in [0,1]$. Hence we have, for any $p \in (0,1)$

$$(1+x)^p = 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n \qquad \forall x \in [0,1] \ .$$

As a by-product, if $p \in (0,1)$, then the series

$$1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!}$$

converges, and

$$1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} = 2^p \qquad \forall p \in (0,1).$$

The proof of

$$(1+x)^p = 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n \qquad \forall x \in [-1,1] \ ,$$

for $p \in (0,1)$ is much more difficult, which will be done in Step 3.
Let us consider the following sequence

$$\begin{aligned} a(p)_n &= \frac{p(p-1)\cdots(p-(n-1))}{n!} \\ &= (-1)^n \frac{(-p)(1-p)\cdots((n-1)-p)}{n!} \ . \end{aligned}$$

If $p \in (0,1)$ then

$$a(p)_n = (-1)^{n-1} \frac{p}{n} \left(1 - \frac{p}{1}\right)\left(1 - \frac{p}{2}\right) \cdots \left(1 - \frac{p}{n-1}\right) \ .$$

Step 2. If $p \in (-1,0)$ then $1+p \in (0,1)$ and we thus may rewrite

$$\begin{aligned} a(p)_n &= (-1)^n \frac{(1-(p+1))(2-(p+1))\cdots(n-(1+p))}{n!} \\ &= (-1)^n \left(1 - \frac{p+1}{1}\right)\left(1 - \frac{p+1}{2}\right) \cdots \left(1 - \frac{p+1}{n}\right) \ . \end{aligned}$$

It is obvious therefore we need to study the sequence

$$\begin{aligned} b(r)_n &= \left(1-\frac{\gamma}{1}\right)\left(1-\frac{\gamma}{2}\right)\cdots\left(1-\frac{\gamma}{n}\right) \\ &= \prod_{k=1}^{n}\left(1-\frac{\gamma}{k}\right) \end{aligned}$$

where γ is a constant. By MVT, for every $t \in (0,\gamma]$

$$\log(1-t) - \log 1 = -\frac{t}{1-\xi}$$

for some ξ between 0 and t, so that

$$-\frac{t}{1-\gamma} \le -\frac{t}{1-t} \le \log(1-t) \le -t \ .$$

Hence

$$e^{-\frac{\gamma}{1-\gamma}\frac{1}{k}} \le \left(1-\frac{\gamma}{k}\right) \le e^{-\frac{\gamma}{k}} \qquad \text{for } k = 1,2,\cdots$$

so that

$$e^{-\frac{\gamma}{1-\gamma}\sum_{k=1}^{n}\frac{1}{k}} \le b(r)_n \le e^{-\gamma\sum_{k=1}^{n}\frac{1}{k}} \qquad \text{for } k = 1,2,\cdots \ . \tag{2.18}$$

In particular, $\lim_{n\to\infty} b(r)_n = 0$, a fact which, together with Taylor's Theorem, allows you to prove that, if $p \in (-1,1)$, then

$$(1+x)^p = 1 + \sum_{n=1}^{\infty}\frac{p(p-1)\cdots(p-(n-1))}{n!}x^n \qquad \forall x \in (-1,1] \ .$$

Step 3. [Difficult Part] If $p \in (0,1)$, then $|a(p)_n| = \frac{p}{n}b(p)_{n-1}$. The inequality (2.18) yields that

$$\begin{aligned} b(p)_{n-1} &\le e^{-p\left(\sum_{k=1}^{n-1}\frac{1}{k}-\log(n-1)\right)+\log(n-1)} \\ &= e^{-p\log(n-1)}e^{-p\left(\sum_{k=1}^{n-1}\frac{1}{k}-\log(n-1)\right)} \\ &= \frac{1}{(n-1)^p}e^{-p\left(\sum_{k=1}^{n-1}\frac{1}{k}-\log(n-1)\right)} \ . \end{aligned}$$

Since, as we have seen before,

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = \gamma$$

exists [Euler's constant], so that

$$e^{-p\left(\sum_{k=1}^{n-1}\frac{1}{k}-\log(n-1)\right)} \le C_1 \ .$$

Hence

$$\begin{aligned} |a(p)_n| &= \frac{p}{n}b(p)_{n-1} \\ &\le C_1 p\frac{1}{n(n-1)^p} \quad \text{for all } \ n \ge 2 \ . \end{aligned}$$

Since $\sum_{n=2}^{\infty} \frac{1}{n(n-1)^p}$ converges for $p > 0$ [Analysis 1, Integral Test], so that

$$\sum_{n=2}^{\infty} |a(p)_n| \leq C_1 p \sum_{n=2}^{\infty} \frac{1}{n(n-1)^p} < +\infty$$

as $p > 0$. It in turn implies that the power series

$$1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n$$

converges absolutely for any $|x| \leq 1$, and in particular it converges uniformly in $[-1, 1]$. Hence for $p > 0$

$$(1+x)^p = 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} x^n \qquad \forall x \in [-1, 1] \ .$$

■

Remark 2.58 *You may use Abel's theorem to handle the end points. In fact, according to Abel's theorem, if*

$$\sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!}$$

converges, then

$$2^p = 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-(n-1))}{n!} \ .$$

Similarly, if (only possible for $p \geq 0$)

$$\sum_{n=1}^{\infty} (-1)^n \frac{p(p-1)\cdots(p-(n-1))}{n!}$$

converges, then

$$1 + \sum_{n=1}^{\infty} (-1)^n \frac{p(p-1)\cdots(p-(n-1))}{n!} = 0 \ .$$

While, you have to use Taylor theorem to verify the convergence of these series.