Professor Joyce Mods Geometry I Lectures MT07

Books: J. Roe, *"Elementary Geometry"*, OUP, 1993.
Or: anything like "vector methods" in your college library.

Lecture 1, Week 1.

1. Vector Geometry

For the first half of this course I'm going to talk about the geometry of **vectors**. Many of you will have met vectors in school, but some of you won't. So we'll start today by defining vectors, and saying some simple things about them.

Geometry happens in a **space** S, dimension $2, 3$ or n.
You can think of S as $\mathbb{R}^n$, the set
$S = \{(x_1, x_2, \ldots, x_n) : x_j \in \mathbb{R}\}$, $\mathbb{R}$ the real numbers.
$(x_1, \ldots, x_n)$ **coordinates** in S.

But this implies a choice of **origin** $(0, \ldots, 0)$ and **axes**.
Better: don't chose origin and axes.
Regard S as featureless, no special points or directions.

Here are three definitions of vectors.
Traditional definition.
"A vector is a quantity with magnitude and direction in S."
Coordinate definition.
A vector is an n-tuple $(v_1, \ldots, v_n)$, $v_j \in \mathbb{R}$.
— addition, subtraction, etc. then defined in coordinates. Often write as **column vectors** $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$.
Vectors as translations.
Let P, Q be points in a space S. Then there is a unique **translation** of S taking P to Q. (A translation is a mapping from S to itself that moves every point the same distance in the same direction.)
Call this translation the **vector** V or $\vec{PQ}$.
— So, vectors, **are** translations of S.
We usually take the last point of view. Write vectors as $\underline{u}, \underline{v}, \ldots$ or $\vec{PQ}$.

1.1. Properties of vectors

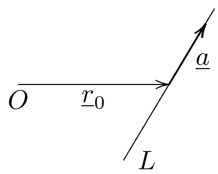
Vector addition: if $\underline{u}, \underline{v}$, are vectors, then $\underline{u} + \underline{v}$ is a vector. $\underline{u} + \underline{v}$ means: do translation $\underline{u}$, then do translation $\underline{v}$. There is a special vector $\underline{0}$, the translation which fixes every point.
We have $\underline{u} + \underline{v} = \underline{v} + \underline{u}$ and $\underline{u} + \underline{0} = \underline{u}$.
Scalar multiplication. If $\lambda \in \mathbb{R}$ and $\underline{u}$ is a vector, then $\lambda\underline{u}$ is a vector.
We have $\lambda(\nu\underline{u}) = (\lambda\nu)\underline{u}$, $0\underline{n} = \underline{0}$,
$1\underline{u} = \underline{u}$, and $\lambda(\underline{u} + \underline{v}) = \lambda\underline{u} + \lambda\underline{v}$.
We write $(-1)\underline{u} = -\underline{u}$.
Then $\underline{u} + (-\underline{u}) = \underline{0}$.

Write $\underline{u} - \underline{v} = \underline{u} + (-\underline{v})$ — vector subtraction.

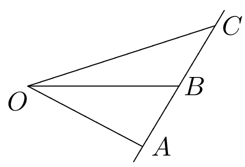
Length of vectors. If $\underline{u}$ is a vector, the length $|\underline{u}|$ is a real number.
$|\underline{u}| \geq 0$, and $|\underline{u}| = 0$ iff $\underline{u} = \underline{0}$.
$|\lambda \underline{u}| = |\lambda||\underline{u}|$ for $\lambda \in \mathbb{R}$.
$|\vec{PQ}|$ is the distance between P and Q in the space S.
In coordinates, $(u_1, \ldots, u_n) + (v_1, \ldots, v_n) = (u_1 + v_1, \ldots, u_n + v_n)$.
$\lambda(u_1, \ldots, u_n) = (\lambda u_1, \ldots, \lambda u_n)$, $\quad |(u_1, \ldots, u_n)| = (u_1^2 + \cdots + u_n^2)^{\frac{1}{2}}$.
Let V be the space (set) of all vectors, the space of all translations of S.
S and V are nearly the same thing. Think of S as V without a choice of origin. Pick an origin O in S. Then any other point P in S is defined by its positive vector $\underline{P} = \vec{OP}$ with respect to O. This defines a 1-1 correspondence between S and V.

1.2. Examples of the use of vectors

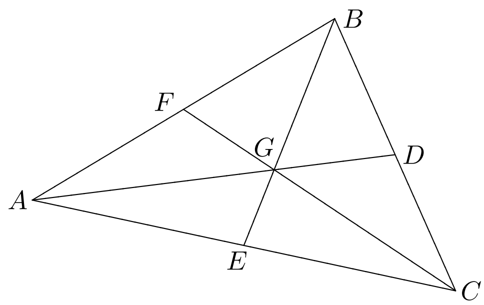
(a) **Equation of a line L.**
If $\underline{r}_0$ is the position vector of a point on L, and $\underline{a}$ is a vector in the direction of L, then $\underline{r}$ is the position vector of a point on L iff $\underline{r} = \underline{r}_0 + t\underline{a}$, some $t \in \mathbb{R}$.



(b) **Proportional division of a line.**
A, B, C on a line L, with position vectors $\underline{a}, \underline{b}, \underline{c}$, and let $\frac{AC}{AB} = t$.
$\vec{AB} = \underline{b} - \underline{a}$, so $\vec{AC} = t(\underline{b} - \underline{a})$. Thus $\vec{OC} = \vec{OA} + \vec{AC} = \underline{a} + t(\underline{b} - \underline{a})$. Hence $\underline{c} = (1-t)\underline{a} + t\underline{b}$.
In particular, when $t = \frac{1}{2}$ the **midpoint** of AB is $\underline{c} = \frac{1}{2}(\underline{a} + \underline{b})$.



(c) **The medians of a triangle.**
ABC is a triangle. D, E, F mid points of BC, CA, AB. The **medians** are AD, BE, CF, the lines joining vertices with midpoints of opposite sides.



Claim: the medians are concurrent, i.e. meet in a point, which is called the *centroid*.

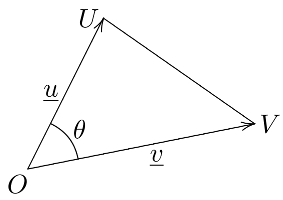
Proof. Let $\underline{a}, \underline{b}, \underline{c}, \ldots$ be position vectors of $A, B, C, \ldots$.
Let G be the point on AO with $\frac{AG}{AD} = \frac{2}{3}$.
Then $\underline{g} = \frac{1}{3}\underline{a} + \frac{2}{3}\underline{d}$, by (b) above.
But $\underline{d} = \frac{1}{2}(\underline{b} + \underline{c})$, by (b) above.
So $\underline{g} = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})$.
By symmetry in $\underline{a}, \underline{b}, \underline{c}$, G also lies on BE, CF. So AD, BE, CF meet in a point. **Q.E.D.**

Note: we have made no assumption about the dimension of S. So this result is true in all dimensions.

Mods Geometry I. Lecture 2, Week 2.

1.3. Lengths and the scalar product.

If $\underline{u}$ is a vector in a space S, it has a length $|\underline{u}|$. In coordinates, if $\underline{u} = (u_1, \ldots, u_n)$ then $|\underline{u}| = (u_1^2 + \cdots + u_n^2)^{\frac{1}{2}}$, so $|\underline{u}|^2 = u_1^2 + \cdots + u_n^2$. Let O, U, V be points in S forming a triangle, with $\underline{u} = \vec{OU}$, $\underline{v} = \vec{OV}$. Let θ be the angle between $\underline{u}$ and $\underline{v}$.



The **cosine rule** says

$$|UV|^2 = |OU|^2 + |OV|^2 - 2|OU||OV|\cos\theta.$$

That is,

$$|\underline{v} - \underline{u}|^2 = |\underline{u}|^2 + |\underline{v}|^2 - 2|\underline{u}||\underline{v}|\cos\theta.$$

Hence

$$|\underline{u}||\underline{v}|\cos\theta = \frac{1}{2}(|\underline{u}|^2 + |\underline{v}|^2 - |\underline{v} - \underline{u}|^2). \qquad (1)$$

Define $\underline{u} \cdot \underline{v} = |\underline{u}||\underline{v}|\cos\theta$. This is called the **scalar product** or **dot product** of $\underline{u}$ and $\underline{v}$. It is a way of multiplying two vectors and getting a real number (scalar).
Note: no restriction on dimension — works in dimension n.
By (1) we have

$$\underline{u} \cdot \underline{v} = \frac{1}{2}(|\underline{u}|^2 + |\underline{v}|^2 - |\underline{v} - \underline{u}|^2).$$

So '$\cdot$' is determined by lengths in S.
In coordinates, if $\underline{u} = (u_1, \ldots, u_n)$, $\underline{v} = (v_1, \ldots, v_n)$, using $|\underline{u}|^2 = u_1^2 + \cdots + u_n^2$, etc., we see that

$$\underline{u} \cdot \underline{v} = u_1v_1 + u_1v_2 + \cdots + u_nv_n. \quad (2) \text{ — Alternative definition of '}\cdot\text{'.}$$

If $\underline{u}, \underline{v} \neq \underline{0}$ but $\underline{u} \cdot \underline{v} = 0$, then $|\underline{u}||\underline{v}|\cos\theta = 0$, $|\underline{u}|, |\underline{v}| \neq 0$ so $\cos\theta = 0$. That is, $\theta = \frac{\pi}{2}$, so $\underline{u}, \underline{v}$ are at right angles. We call two vectors $\underline{u}, \underline{v}$ with $\underline{u} \cdot \underline{v} = 0$ **orthogonal**, or **perpendicular**.

Proposition. *Let $\underline{u}, \underline{v}, \underline{w}$ be vectors, and $\lambda \in \mathbb{R}$. Then*

(i) $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$ *"symmetric"*

(ii) $(\lambda\underline{u}) \cdot \underline{v} = \lambda(\underline{u} \cdot \underline{v})$ *"bilinear"*

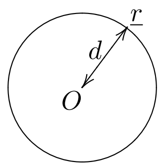
(iii) $\underline{u} \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$ *"bilinear"*

(iv) $\underline{u} \cdot \underline{u} = |\underline{u}|^2 \geq 0$, *and* $\underline{u} \cdot \underline{u} = 0$ *iff* $\underline{u} = \underline{0}$. *"positive definite"*

— Here (i)-(iii) follow from (2), and (iv) from (1). Parts (i)-(iii) mean we can treat '$\cdot$' like ordinary multiplication — multiply out brackets, and so on.

1.4. Examples in 2 dimensions.

(a) The equation of a circle.

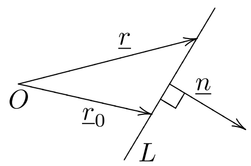


$\underline{r}$ is the position vector of a point on the circle with center $\underline{c}$, radius d, iff $|\underline{r}-\underline{c}| = d$. Equivalently,

$|\underline{r}-\underline{c}|^2 = \underline{d}^2 \Leftrightarrow \quad (\underline{r}-\underline{c})\cdot(\underline{r}-\underline{c}) = d^2 \Leftrightarrow$
$\underline{r}\cdot\underline{r} - 2\underline{r}\cdot\underline{c} + \underline{c}\cdot\underline{c} = d^2$ (3) equation of a circle.

Exercise: put $\underline{r} = (x, y)$ and expand in (x, y) coordinates.

(b) The equation of a line. Let L be a line. Let $\underline{r}_0$ be the position vector of a



point on L. Let $\underline{n}$ be a vector perpendicular to L. Can choose $|\underline{n}| = 1$, so $\underline{n}$ is a **unit vector**.
Then $\underline{r}$ lies on L iff

$\underline{r}-\underline{r}_0$ is parallel to $L \Leftrightarrow$

$\underline{r}-\underline{r}_0$ is perpendicular to $\underline{n} \Leftrightarrow$

$(\underline{r}-\underline{r}_0)\cdot\underline{n} = 0.$

So, equation of a line in $\mathbb{R}^2$ is $\underline{r}\cdot\underline{n} = \underline{r}_0\cdot\underline{n}$. (4)

Note: In 3 dimensions, (3) and (4) are the equations of a **sphere** and of a **plane** in $\mathbb{R}^3$.

2. Three-dimensional geometry and the vector product

In this section we work strictly in 3 dimensions. Given two vectors $\underline{u}, \underline{v}$ we are going to define another vector $\underline{u}\wedge\underline{v}$ (or $\underline{u}\times\underline{v}$) (pronounced $\underline{u}$ vec $\underline{v}$ or $\underline{u}$ cross $\underline{v}$) called the **vector product** or **cross product** of $\underline{u}$ and $\underline{v}$.

learn this $\begin{cases} \text{coordinate definition: } \underline{u} = (u_1, u_2, u_3), \quad \underline{v} = (v_1, v_2, v_3) \\ (u_1, u_2, u_3)\wedge(v_1, v_2, v_3) = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1). \end{cases}$

Equivalently: write $\underline{i} = (1, 0, 0)$, $\underline{j} = (0, 1, 0)$, $\underline{k} = (0, 0, 1)$. Then $(u_1, u_2, u_3) = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$, $(v_1, v_2, v_3) = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$ and

$$\left.\begin{array}{l} \underline{i}\wedge\underline{j} = \underline{k}, \quad \underline{j}\wedge\underline{k} = \underline{i}, \quad \underline{k}\wedge\underline{i} = \underline{j} \\ \underline{j}\wedge\underline{i} = -\underline{k}, \quad \underline{k}\wedge\underline{j} = -\underline{i}, \quad \underline{i}\wedge\underline{k} = -\underline{j} \\ \text{and } \underline{i}\wedge\underline{i} = \underline{j}\wedge\underline{j} = \underline{k}\wedge\underline{k} = \underline{0}. \end{array}\right] \quad \text{learn these}$$

We can also write $\underline{u} \wedge \underline{v}$ as a determinant:

$$\underline{u} \wedge \underline{v} = \begin{vmatrix} \underline{i} & u_1 & v_1 \\ \underline{j} & u_2 & v_2 \\ \underline{k} & u_3 & v_3 \end{vmatrix} \quad \text{— this might help you to remember the formula.}$$

Proposition 1.

(i) $\underline{u} \wedge \underline{v} = \underline{v} \wedge \underline{u}$, *and* $\underline{u} \wedge \underline{u} = \underline{0}$ *(antisymmetry)*

(ii) $(\alpha\underline{u} + \beta\underline{v}) \wedge \underline{w} = \alpha\underline{u} \wedge \underline{w} + \beta\underline{v} \wedge \underline{w}$ *(bilinearity)*

(iii) $\underline{u} \cdot (\underline{u} \wedge \underline{v}) = \underline{v} \cdot (\underline{u} \wedge \underline{v}) = 0$, *that is,* $\underline{u}, \underline{v}$ *are perpendicular to* $\underline{u} \wedge \underline{v}$.

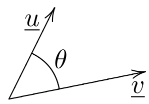
— all immediate from the definition.

Mods Geometry I. Lecture 3, Week 2.

— Continuing section 2 on the vector product. Work strictly in 3 dimensions.

2.1. The length of the vector product

Proposition 2. $|\underline{u} \wedge \underline{v}| = |\underline{u}||\underline{v}| \sin\theta$, *where* θ *is the angle between* $\underline{u}$ *and* $\underline{v}$.



Proof. By multiplying out we find that

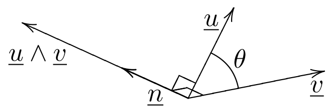
$$(u_1v_1 + u_2v_2 + u_3v_3)^2 + (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2$$
$$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2).$$

This implies that $(\underline{u} \cdot \underline{v})^2 + |\underline{u} \wedge \underline{v}|^2 = |\underline{u}|^2|\underline{v}|^2$.

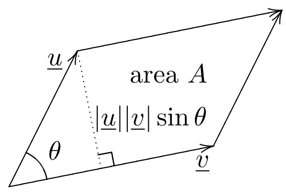
Since $\underline{u} \cdot \underline{v} = |\underline{u}||\underline{v}| \cos\theta$ and $\cos^2\theta + \sin^2\theta = 1$, we have $|\underline{u} \wedge \underline{v}|^2 = |\underline{u}|^2|\underline{v}|^2 \sin^2\theta$.

As $\theta \in [0, \pi]$, $\sin\theta \geq 0$, so equation follows by square roots. **Q.E.D.**

Combining Proposition 1(iii) and Proposition 2 we see that $\underline{u} \wedge \underline{v} = |\underline{u}||\underline{v}| \sin\theta\, \underline{n}$, where θ is the angle between $\underline{u}$ and $\underline{v}$, and $\underline{n}$ is a unit vector perpendicular to $\underline{u}, \underline{v}$.



Equivalently, $\underline{u} \wedge \underline{v} = A\, \underline{n}$, $A = |\underline{u}||\underline{v}| \sin\theta$ area of parallelogram with sides $\underline{u}, \underline{v}$.



Remark: There are two choices for $\underline{n}$, so there is a problem in choosing the sign of $\underline{n}$, i.e. which of two the unit normals to plane $\langle \underline{u}, \underline{v} \rangle$ to take. An **orientation** is a way of making this choice — it distinguishes between right handed and left handed. The **right hand rule** gives direction of $\underline{u} \wedge \underline{v}$, and sign of $\underline{n}$: take $\underline{u}$ along thumb, $\underline{v}$ along first finger, $\underline{u} \wedge \underline{v}$ along middle finger of right hand.

2.2. The scalar triple product

Given 3 vectors $\underline{u}, \underline{v}, \underline{w}$ in $\mathbb{R}^3$, define the **scalar triple product** $[\underline{u}, \underline{v}, \underline{w}] = \underline{u} \cdot (\underline{v} \wedge \underline{w})$. In coordinates, with $\underline{u} = (u_1, u_2, u_3)$, $\underline{v} = (v_1, v_2, v_3)$, $\underline{w} = (w_1, w_2, w_3)$,

$$(\underline{u}, \underline{v}, \underline{w}) = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1)$$

$$(1) \qquad = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

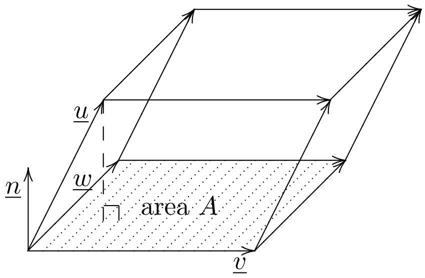
(This is a **determinant**. If you don't know about determinants, ask your college tutor.)

Proposition.

(i) $[\underline{u}, \underline{v}, \underline{w}] = [\underline{v}, \underline{w}, \underline{u}] = [\underline{w}, \underline{u}, \underline{v}] = -[\underline{u}, \underline{w}, \underline{v}] = -[\underline{v}, \underline{u}, \underline{w}] = -[\underline{w}, \underline{v}, \underline{u}]$,

(ii) $[\underline{u}, \underline{v}, \underline{w}] = 0$ *iff* $\underline{u}, \underline{v}, \underline{w}$ *are* **coplanar** *(i.e. lie in a plane).*

Proof. (i) follows from (1), and (ii) from properties of determinants, or geometric interpretation below. **Q.E.D.**



Geometric interpretation of the scalar triple product.
Let $\underline{u}, \underline{v}, \underline{w}$ be vectors. Let Π be the plane containing $\underline{v}$ and $\underline{w}$, and A the area of the parallelogram in Π with sides $\underline{v}, \underline{w}$. Let $\underline{n}$ be a unit normal to Π. Then $\underline{v} \wedge \underline{w} = A\underline{n}$, from section 2.1. So $\underline{u} \cdot (\underline{v} \wedge \underline{w}) = A(\underline{u} \cdot \underline{n})$.
But $\underline{u} \cdot \underline{n}$ is the height of the parallelepiped with edges $\underline{u}, \underline{v}, \underline{w}$, measured from the plane Π. Volume of parallelepiped = (area of base)$\times$(height) = $A(\underline{u} \cdot \underline{n})$.
Thus, $\big|[\underline{u}, \underline{v}, \underline{w}]\big|$ is the **volume** of the parallelepiped with edges $\underline{u}, \underline{v}, \underline{w}$.
This volume is zero iff $\underline{u}, \underline{v}, \underline{w}$ are coplanar, giving (ii) above.

2.3. The vector triple product

Given vectors $\underline{u}, \underline{v}, \underline{w}$ in $\mathbb{R}^3$, their **vector triple product** is $\underline{u} \wedge (\underline{v} \wedge \underline{w})$. This is orthogonal to $\underline{v} \wedge \underline{w}$ by properties of '$\wedge$'. So it lies in the plane containing $\underline{v}$ and $\underline{w}$. Hence $\underline{u} \wedge (\underline{v} \wedge \underline{w}) = \alpha\underline{v} + \beta\underline{w}$, some $\alpha,\ \beta \in \mathbb{R}$.

Proposition. $\underline{u} \wedge (\underline{v} \wedge \underline{w}) = (\underline{u} \cdot \underline{w})\underline{v} - (\underline{u} \cdot \underline{v})\underline{w}$ — learn this formula. Rule is "outside pair first".

Proof. Expand out in coordinates. We'll use vectors $\underline{i},\ \underline{j},\ \underline{k}$. To save work, choose axes so that $\underline{u} = u_1\underline{i}$, and write

$$\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$$
$$\underline{w} = w_1\underline{i} + w_2\underline{j} + w_3\underline{k}.$$

Then

$$\begin{aligned}\underline{u} \wedge (\underline{v} \wedge \underline{w}) &= u_1\underline{i} \wedge ((v_2w_3 - v_3w_2)\underline{i} + (v_3w_1 - v_1w_3)\underline{j} + (v_1w_2 - v_2w_1)\underline{k}) \\ &= u_1(v_3w_1 - v_1w_3)\underline{k} - u_1(v_1w_2 - v_2w_1)\underline{j} \\ &= u_1w_1(v_2\underline{j} + v_3\underline{k}) - u_1v_1(w_2\underline{j} + w_3\underline{k}) \\ &= u_1w_1(\underline{v} - \cancel{v_1\underline{i}}) - u_1v_1(\underline{w} - \cancel{w_1\underline{i}}) \\ &= (\underline{u} \cdot \underline{w})\underline{v} - (\underline{u} \cdot \underline{v})\underline{w}.\end{aligned}$$

Q.E.D.

Note: $\underline{u} \wedge (\underline{v} \wedge \underline{w}) \neq (\underline{u} \wedge \underline{v}) \wedge \underline{w}$. That is, '$\wedge$' is **not associative**. So, expressions like $\underline{u} \wedge \underline{v} \wedge \underline{w}$ make no sense; need brackets to tell you which operation to do first.

Mods Geometry I. Lecture 4, Week 3.

2.4. The equation of a line

Consider the equation $\underline{r} \wedge \underline{a} = \underline{b}$ in 3 dimensions for $\underline{a} \neq \underline{0}$.
What are the solutions for $\underline{r}$?
Take scalar product with $\underline{a}$:

$$0 = \underline{a} \cdot (\underline{r} \wedge \underline{a}) = \underline{a} \cdot \underline{b}.$$

So there are no solutions unless $\underline{a} \cdot \underline{b} = 0$.
Vector product with $\underline{a}$:

$$\underline{a} \wedge (\underline{r} \wedge \underline{a}) = (\underline{a} \cdot \underline{a})\underline{r} - (\underline{a} \cdot \underline{r})\underline{a} = \underline{a} \wedge \underline{b}.$$

Hence

$$\underline{r} = \frac{\underline{a} \wedge \underline{b}}{|\underline{a}|^2} + t\underline{a},$$

where

$$t = \frac{\underline{a} \cdot \underline{r}}{|\underline{a}|^2}.$$

Conversely, if $\underline{a} \cdot \underline{b} = 0$, substitution shows that every element of this form is a solution to $\underline{r} \wedge \underline{a} = \underline{b}$.
Thus, we have the following:

Proposition. *If $\underline{a}$, $\underline{b}$ are vectors in $\mathbb{R}^3$ with $\underline{a} \neq \underline{0}$ and $\underline{a} \cdot \underline{b} = 0$, then $\underline{r} \wedge \underline{a} = \underline{b}$ iff $\underline{r} = \frac{\underline{a} \wedge \underline{b}}{|\underline{a}|^2} + t\underline{a}$, for $t \in \mathbb{R}$.*

Now $\underline{r} = \frac{\underline{a} \wedge \underline{b}}{|\underline{a}|^2} + t\underline{a}$ is the *equation of a line* parallel to $\underline{a}$, passing through $\frac{\underline{a} \wedge \underline{b}}{|\underline{a}|^2}$.
Hence, $\underline{r} \wedge \underline{a} = \underline{b}$ is the equation of a line. All lines in $\mathbb{R}^3$ can be written this way.

2.5. Examples

(a) Let Π_1: $\underline{r} \cdot \underline{n}_1 = d_1$, Π_2: $\underline{r} \cdot \underline{n}_2 = d_2$ be two non-parallel planes in $\mathbb{R}^3$. Find the equation of the line $\Pi_1 \cap \Pi_2$ in the form $\underline{r} \wedge \underline{a} = \underline{b}$.

Solution: The line is parallel to $\underline{a}$, so $\underline{a}$ is parallel to Π_1, Π_2, and thus perpendicular to $\underline{n}_1, \underline{n}_2$.
Therefore, $\underline{a}$ is parallel to $\underline{n}_1 \wedge \underline{n}_2$, which is non-zero as Π_1, Π_2 are non-parallel, so $\underline{n}_1, \underline{n}_2$ are non-parallel.
Take $\underline{a} = \underline{n}_1 \wedge \underline{n}_2$. Then

$$\begin{aligned} \underline{r} \wedge \underline{a} &= \underline{r} \wedge (\underline{n}_1 \wedge \underline{n}_2) \\ &= (\underline{r} \cdot \underline{n}_2)\underline{n}_1 - (\underline{r} \cdot \underline{n}_1)\underline{n}_2 \\ &= d_2\underline{n}_1 - d_1\underline{n}_2. \end{aligned}$$

Thus, $\underline{r} \wedge (\underline{n}_1 \wedge \underline{n}_2) = d_2\underline{n}_1 - d_1\underline{n}_2$ is the equation we want.

(b) Let L be the line $\underline{r} \wedge \underline{a} = \underline{b}$, where $\underline{a} \cdot \underline{b} = 0$. Let Π be the plane $\underline{r} \cdot \underline{c} = \alpha$.
Suppose $\underline{a} \cdot \underline{c} \neq 0$, so that L is not parallel to Π.

Claim: $L \cap \Pi$ is a single point.

Proof:

$$\underline{r} = \frac{\underline{a} \wedge \underline{b}}{|\underline{a}|^2} + t\underline{a}, \quad \text{for} \quad \underline{r} \in L.$$

Substitute into $\underline{r} \cdot \underline{c} = \alpha$:

$$\frac{[\underline{a}, \underline{b}, \underline{c}]}{|\underline{a}|^2} + t(\underline{a} \cdot \underline{c}) = \alpha,$$

and so

$$t = \frac{\alpha}{\underline{a} \cdot \underline{c}} - \frac{[\underline{a}, \underline{b}, \underline{c}]}{(\underline{a} \cdot \underline{c})|\underline{a}|^2}.$$

This gives us a unique point $\underline{r}$ in $L \cap \Pi$.

3. Isometries

Let S be an n-dimensional space. An *isometry* of S is a map $T : S \to S$, which is a bijection (1–1 and onto), and *preserves distances*; i.e., $|T(A)\,T(B)| = |AB|$ for $A, B \in S$. *Rotations*, *reflections* and *translations* are examples of isometries.

3.1. Action of isometries on vectors

Let S be an n-dimensional space. Let V be the set of vectors in S; i.e., each $\underline{v} \in V$ is a *translation*, $\underline{v} : S \to S$. Let $T : S \to S$ be an isometry.

Proposition 1. *The map $T \circ \underline{v} \circ T^{-1} : S \to S$ is a translation of S, and hence a vector in V. Define $\overrightarrow{T}\underline{v} = T \circ \underline{v} \circ T^{-1}$. Then $\overrightarrow{T} : V \to V$ is a linear map.*

Proof. Easy (Roe, p. 79–80), but we won't give it.
If $\underline{v}$ takes A to B in S, so $\underline{v} = \overrightarrow{AB}$, then $\overrightarrow{T}\underline{v}$ takes $T(A)$ to $T(B)$, so $\overrightarrow{T}\underline{v} = \overrightarrow{T(A)\,T(B)}$.
But $|T(A)\,T(B)| = |AB|$ by definition. Hence, $|\overrightarrow{T}\underline{v}| = |\underline{v}|$ for all $v \in V$.
Now, from section 1.3,

$$\underline{u} \cdot \underline{v} = \frac{1}{2}\left(|\underline{u}|^2 + |\underline{v}|^2 - |\underline{v} - \underline{u}|^2\right),$$

so we see that $(\overrightarrow{T}\underline{u}) \cdot (\overrightarrow{T}\underline{v}) = \underline{u} \cdot \underline{v}$, i.e., $\overrightarrow{T}$ preserves the scalar product.

3.2. Isometries in coordinates

Now choose an origin O in S, and represent each point R in S by its position vector $\underline{r} = \overrightarrow{OR}$.

Proposition 2. *Every isometry T can be written as $T(\underline{r}) = \overrightarrow{T}\underline{r} + \underline{b}$ on position vectors, where $\overrightarrow{T}$ is a linear map on vectors with $(\overrightarrow{T}\underline{u}) \cdot (\overrightarrow{T}\underline{v}) = \underline{u} \cdot \underline{v}$, and $\underline{b}$ is a fixed vector.*

Proof. As $\underline{r} = \overrightarrow{OR}$,

$$\begin{aligned} T(\underline{r}) &= \overrightarrow{O\,T(R)} \\ &= \overrightarrow{T(O)\,T(R)} + \overrightarrow{O\,T(O)} \\ &= \overrightarrow{T}(\overrightarrow{OR}) + \underline{b} \\ &= \overrightarrow{T}\underline{r} + \underline{b}, \end{aligned}$$

where $\underline{b} = \overrightarrow{O\,T(O)}$. The rest follows from above. **Q.E.D.**

Definition. An $n \times n$ matrix A is called *orthogonal* if $A^t A = I$, where A^t is the transpose of A, and I the $n \times n$ identity matrix.

Theorem. *Let T be an isometry of $\mathbb{R}^n$, and write points as column vectors $\underline{x} = (x_1, \ldots, x_n)^t$. Then T acts as $T\underline{x} = A\underline{x} + \underline{b}$, where A is an $n \times n$ orthogonal matrix and $\underline{b} = (b_1, \ldots, b_n)^t$. Conversely, every transformation of this form is an isometry.*

Proof. Linear maps on column vectors act by matrices, so in Proposition 2, $\overrightarrow{T}\underline{x} = A\underline{x}$ for some matrix A. Also,

$$(\overrightarrow{T}\underline{u}) \cdot (\overrightarrow{T}\underline{v}) = \underline{u} \cdot \underline{v} \quad \Leftrightarrow \quad (A\underline{u})^t(A\underline{v}) = \underline{u}^t\underline{v},$$

in matrix notation, i.e.,

$$\begin{aligned} & \underline{u}^t A^t A\underline{v} = \underline{u}^t\underline{v} \\ \Rightarrow \quad & \underline{u}^t(A^tA - I)\underline{v} = 0, \end{aligned}$$

for all $\underline{u}, \underline{v}$ in $\mathbb{R}^n$.
This happens iff $A^tA = I$, so A is orthogonal, proving the first part. For the converse note that

$$\begin{aligned} |T\underline{x}\ T\underline{y}|^2 &= |A(\underline{y} - \underline{x})|^2 \\ &= (\underline{y} - \underline{x})^t A^t A(\underline{y} - \underline{x}) \\ &= (\underline{y} - \underline{x})^t(\underline{y} - \underline{x}) \\ &= |\underline{y} - \underline{x}|^2 \\ &= |\underline{x}\ \underline{y}|^2, \end{aligned}$$

so T is an isometry. **Q.E.D.**

Mods Geometry I. Lecture 5, Week 3.

3. Isometries, continued

Recall: An isometry $T : \mathbb{R}^n \to \mathbb{R}^n$ may be written $T\underline{x} = A\underline{x} + \underline{b}$, A orthogonal $n \times n$ matrix, $\underline{x} = (x_1, \ldots, x_n)^t$.

Proposition. *Let A be an orthogonal $n \times n$ matrix, then* $\det A = \pm 1$.

Proof. $A^t A = I$, so $\det(A^t)\det A = \det I = 1$. But $\det(A^t) = \det(A)$, so $(\det A)^2 = 1$. **Q.E.D.**

This divides isometries into two kinds: those with $\det A = 1$ (often rotations) and those with $\det A = -1$ (often reflections).

3.3. Two-dimensional orthogonal matrices

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 orthogonal matrix. Then $A^t A = I$, so

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

giving the equations

$$\begin{aligned} a^2 + c^2 &= 1 \\ b^2 + d^2 &= 1 \\ ab + cd &= 0 \end{aligned}$$

$$\begin{aligned} \text{First equation} \Rightarrow &\quad a = \cos\theta, \quad c = \sin\theta. \\ \text{Second equation} \Rightarrow &\quad d = \cos\phi, \quad b = -\sin\phi. \\ \text{Third equation} \Rightarrow &\quad -\cos\theta\sin\phi + \sin\theta\cos\phi = 0. \\ &\text{i.e., } \sin(\theta - \phi) = 0. \end{aligned}$$

So $\phi = \theta$ or $\theta + \pi$.

Case (i): $\phi = \theta$, $\quad A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Rotation by angle θ. Note that $\det A = 1$.

Case (ii) $\phi = \theta + \pi$, $\quad A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

This is a *reflection* in line parallel to $(\cos(\theta/2), \sin(\theta/2))^t$. Note that $\det A = -1$.

So orthogonal 2×2 matrices A are either *rotations* ($\det A = 1$) or *reflections* ($\det A = -1$).

3.4. Examples of 3×3 orthogonal matrices

(a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$,

which is a rotation by angle θ about $(1,0,0)^t = x$-axis.

Note that $\det A = 1$, $\operatorname{trace} A = 1 + 2\cos\theta$, $A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Case $\theta = 0$ gives $A = I$, identity.

(b) $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$,

which is a combination of rotation by θ about the x-axis and reflection in the y, z plane.

Note that $\det A = -1$, $\operatorname{trace} A = -1 + 2\cos\theta$, $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Case $\theta = 0$, $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which is *reflection* in the y, z plane.

Case $\theta = \pi$, $A = -I$.

It turns out that all 3×3 orthogonal matrices A are *either* a rotation about some axis, as in (a), *or* the combination of a rotation about some axis, and a reflection perpendicular to the same axis, as in (b).

Note: Some A are neither rotations nor reflections, as in (b) with $\theta \neq 0$.

3.5. Rotations in $\mathbb{R}^3$

Let A be an $n \times n$ matrix. If $A\underline{v} = \lambda\underline{v}$ for $\lambda \in \mathbb{R}$ and $\underline{v} \neq 0$ in $\mathbb{R}^n$, then λ is called an *eigenvalue* of A, and $\underline{v}$ an *eigenvector* of A. In the Linear Algebra lectures in Hilary term you will learn about eigenvalues and eigenvectors.

Let A be a 3×3 orthogonal matrix, with $\det A = 1$. Then one can show that the eigenvalues of A are $1, \lambda, \bar{\lambda}$ for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. So 1 is an eigenvalue of A. That is,

Proposition. *Let A be a 3×3 orthogonal matrix with $\det A = 1$. Then there exists a unit vector $\underline{x}$ in $\mathbb{R}^3$ with $A\underline{x} = \underline{x}$.*

We use this to prove:

Theorem. *Let A be a 3×3 orthogonal matrix with $\det A = 1$. Then A is a rotation by angle θ about some vector $\underline{x}$, where $1 + 2\cos\theta = \operatorname{trace}(A)$.*

Proof. Let $\underline{x}$ be as in the proposition above.

Choose an orthogonal matrix B with $B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \underline{x}$.

Put $C = B^t AB$. Then C is orthogonal, and

$$C \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = B^t AB \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = B^t A\underline{x} = B^t\underline{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$.

Now $\det C = (\det B)^2 \det A = 1$, as $\det B = \pm 1$.

Thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is orthogonal and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$.

So $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ from Section 3.3 (Two-Dimensional Orthogonal Matrices).

So $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ is rotation by θ about $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\operatorname{trace}(C) = 1 + 2\cos\theta$.

Therefore, A is rotation by θ about $\underline{x}$, and

$$\operatorname{trace}(C) = \operatorname{trace}(B^t(AB)) = \operatorname{trace}(ABB^t) = \operatorname{trace}(A),$$

and so $\operatorname{trace}(A) = 1 + 2\cos\theta$. **Q.E.D.**

Useful test: Let A be a 3×3 orthogonal matrix, then A is a *rotation* by angle θ iff $\det A = 1$ and $\operatorname{trace}(A) = 1 + 2\cos\theta$. We can also show that A is a *reflection* iff $\det A = -1$ and $\operatorname{trace}(A) = 1$.

Mods Geometry I. Lecture 6, Week 4.

4. Curves and Surfaces

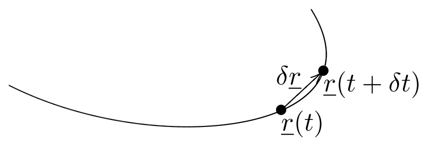
4.1. Curves

A *curve* in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a smooth 1-dimensional subset.
There are two main ways to define curves:

(a) for curves in the plane $\mathbb{R}^2$: as the zeroes of a function $f(x, y) = 0$.

Example: $x^2 + y^2 - r^2 = 0$ is a circle of radius r about $(0, 0)$.

(b) in *parametric form*, as the image of a function $t \mapsto \underline{r}(t)$ for t in some interval (a, b). We call t a *parameter*. In many applications, t is time, $\underline{r}(t)$ is the position vector of a moving point at time t, and C is the path of the point.
In $\mathbb{R}^3$, we can write $\underline{r}(t) = (x(t), y(t), z(t))$, where x, y, z are functions of t.
Suppose the curve is *smooth*. (For now, this means the functions $x(t), y(t), \ldots$ have continuous derivatives.) Consider two neighbouring points on it, parametrized by t and $t + \delta t$. The vector $\delta \underline{r} = \underline{r}(t + \delta t) - \underline{r}(t)$ is a straight-line approximation to the



curve. As $\delta t \to 0$, this vector becomes *tangential* to the curve. We define

$$\frac{d\underline{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta \underline{r}}{\delta t}.$$

In components, $\dfrac{d\underline{r}}{dt} = \left(\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt} \right)$.

If t is time, $\dfrac{d\underline{r}}{dt}$ is the *velocity vector* of the point $\underline{r}(t)$.

The unit vector $t = \dfrac{d\underline{r}}{dt} \Big/ \left| \dfrac{d\underline{r}}{dt} \right|$ is the *unit tangent* to C at $\underline{r}(t)$.

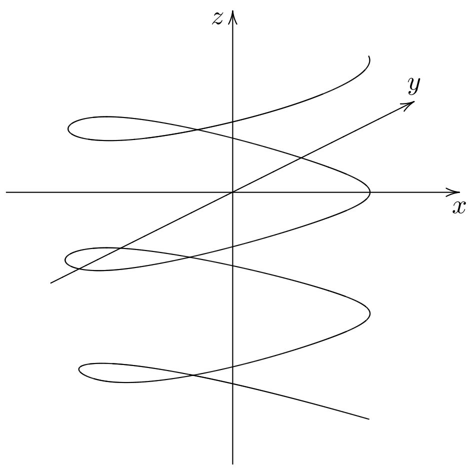
The length of the line segment from $\underline{r}(t)$ to $\underline{r}(t + \delta t)$ is $\delta s = |\delta \underline{r}| \approx \left| \dfrac{d\underline{r}}{dt} \right| \delta t$.

Splitting the curve up into small segments and adding their lengths together, the *length* of the curve from $\underline{r}(t_0)$ to $\underline{r}(t_1)$ is

$$L = \int_{t_0}^{t_1} \left| \frac{d\underline{r}}{dt} \right| dt.$$

(We can show that this is independent of choice of parametrization, using the *chain rule*.)
Often one writes s for *arclength* measured along the curve, and uses s as a parameter. Then $\underline{t} = \frac{d\underline{r}}{ds}$ is the unit tangent.
For a curve C in $\mathbb{R}^2$, the *unit normal* $\underline{n}$ at $\underline{r}(t)$ in C is a unit vector perpendicular to the unit tangent $\underline{t}$.
If C is defined by $f(x, y) = 0$, then $\left(\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right)$ is parallel to the unit normal at (x, y).

Example: The curve $\underline{r} = (a\cos t, a\sin t, bt)$ is a helix in $\mathbb{R}^3$. Find the length of one complete turn, corresponding to $0 \leq t \leq 2\pi$.



Answer: The length is given by

$$\begin{aligned}\int_0^{2\pi} \left|\frac{d\underline{r}}{dt}\right| dt &= \int_0^{2\pi} |(-a\sin t, a\cos t, b)| dt \\ &= \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} dt \\ &= 2\pi\sqrt{a^2 + b^2}.\end{aligned}$$

4.2. Surfaces in $\mathbb{R}^3$

A surface S in $\mathbb{R}^3$ is a smooth 2-dimensional subset. There are two main ways to define surfaces:

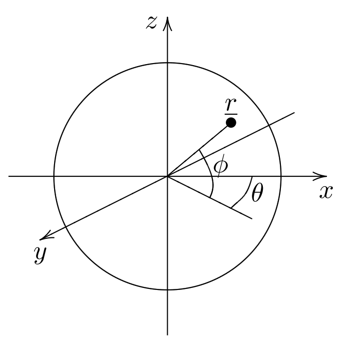
(a) as the zeroes of a function $f(x, y, z) = 0$.

(b) as the image of a function $(s, t) \mapsto \underline{r}(s, t)$, for $(s, t) \in [a, b] \times [c, d]$.

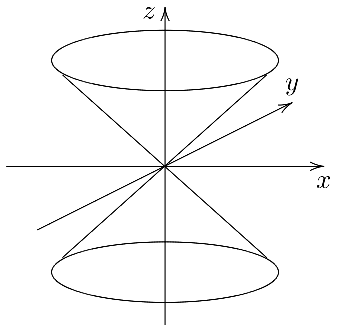
Example: Let S be the sphere of radius r about (0,0,0) in $\mathbb{R}^3$. Then we can write S as $f(x, y, z) = x^2 + y^2 + z^2 - r^2 = 0$, and also as the image of the function

$$(\theta, \phi) \mapsto (r\cos\theta\cos\phi, r\sin\theta\cos\phi, r\sin\phi) = \underline{r}(\theta, \phi)$$

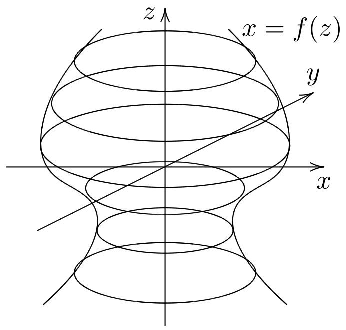
for $(\theta, \phi) \in [0, 2\pi] \times [-\pi/2, \pi/2]$, using *spherical polar coordinates.*



Example: The *(right circular) cone* in $\mathbb{R}^3$ is the surface $x^2+y^2=z^2$. *Conics* (next lecture) are the curves made by the intersection of the cone and a plane in $\mathbb{R}^3$.



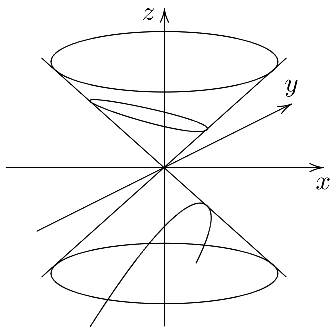
Example: *A surface of revolution* is a surface symmetric under rotations about the z-axis. Starting with a curve $x=f(z), y=0$ in $\mathbb{R}^3$, we can make a surface of revolution by revolving about the z-axis, to get $x^2+y^2=f(z)^2$. Cones and spheres are both surfaces of revolution.



Mods Geometry I. Lecture 7, Week 4.

5. Conics

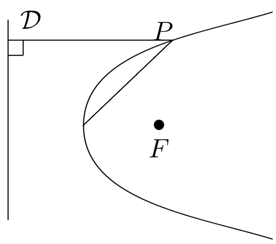
The *conics*, or *conic sections*, are a family of curves in the plane $\mathbb{R}^2$.
One way to define them is as the intersection of the cone $x^2 + y^2 = z^2$ in $\mathbb{R}^3$ with a plane—hence 'conic section'.



See the handout on conics for this lecture, at the end of these lecture notes.

5.1. Focus-directrix definition of conics

Let $\mathcal{D}$ be a line in the plane. Let F be a point, not on $\mathcal{D}$. Let $e > 0$ be a constant. We define the curve C to be the locus of all points P in the plane such that $|PF| = e|P\mathcal{D}|$, where $|P\mathcal{D}|$ is the perpendicular distance from P to $\mathcal{D}$.

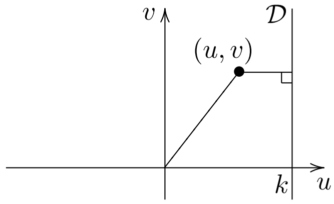


We call C a *conic*, the line $\mathcal{D}$ the *directrix*, the point F the *focus*, and constant e the *eccentricity*.

Note: This definition does not work for circles.

5.2. Equations in Cartesian coordinates

Start with coordinates (u, v), $f = (0, 0)$, $\mathcal{D}$ is $u = k$, $k > 0$.



Then

$$|P\mathcal{D}| = |u - k|, \quad |PF|^2 = u^2 + v^2.$$

So equation of conic is

$$u^2 + v^2 = e^2(u - k)^2,$$

that is, $(1 - e^2)u^2 + 2ke^2u + v^2 = k^2e^2$.
Change to more convenient coordinates:

Case $e \neq 1$: Put $x = u + \dfrac{ke^2}{1 - e^2}$, $y = v$.
We get $(1 - e^2)x^2 + y^2 = \dfrac{ke^2}{1 - e^2}$.
If $0 < e < 1$, this is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

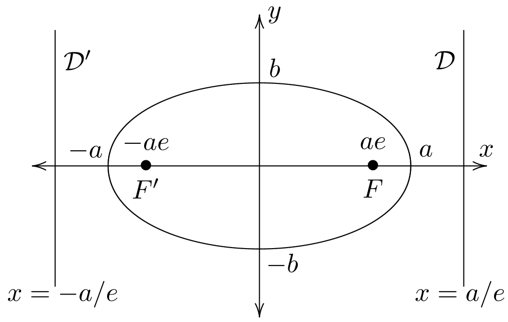
where $a = \dfrac{ke}{1 - e^2}$ and $b = \dfrac{ke}{\sqrt{1 - e^2}}$, which is an *ellipse*.
If $e > 1$, this is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $a = \dfrac{ke}{e^2 - 1}$ and $b = \dfrac{ke}{\sqrt{e^2 - 1}}$, which is a *hyperbola*.
In both cases, the focus is $(ae, 0)$, and the directrix is $x = a/e$.

Two other cases:
The *circle*: $x^2 + y^2 = a^2$, $e = 0$, focus $(0, 0)$, no directrix.
The *parabola*: $y^2 = 4ax$, focus $(a, 0)$, directrix $x = -a$.
See the handout for more information.

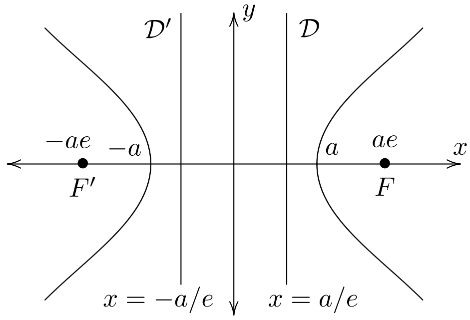
Consider the *ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
It has focus $(ae, 0)$ and directrix $x = a/e$. However, the curve is symmetric under reflection in the y-axis. $(x, y) \mapsto (-x, y)$.
So we can define the *same* ellipse with focus $F' = (-ae, 0)$, and directrix $\mathcal{D}$ given by $x = -a/e$. So an ellipse has two foci and two directrices.



If $P = (x, y)$ lies in C, then $|PF| = e|P\mathcal{D}| = e(\frac{a}{e} - x)$.
Also, $|PF'| = e|P\mathcal{D}'| = e(x + \frac{a}{e})$.
Adding, $|PF| + |PF'| = 2a$.
This proves:

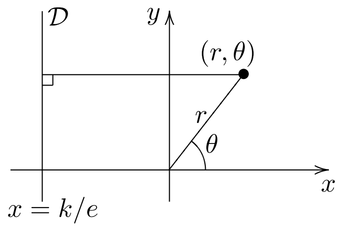
Proposition. *The ellipse with eccentricity* $0 < e < 1$ *and foci* $F = (ae, 0)$ *and* $F' = (-ae, 0)$ *is the locus of points* P *with* $|PF| + |PF'| = 2a$.

A similar result holds for the *hyperbola*: it has focus $F = (ae, 0)$, directrix $\mathcal{D} : x = a/e$, and also $F' = (-ae, 0)$, directrix $\mathcal{D}' : x = -a/e$, and is the set of points P with $\big||PF| - |PF'|\big| = 2a$.



5.3. Equations of conics in polar coordinates

Use polar coordinates (r, θ), with $x = r\cos\theta$, and $y = r\sin\theta$.
Let the focus be $F = (0, 0)$, and the directrix $\mathcal{D} : x = -k/e$. If $P = (r, \theta)$ in polar



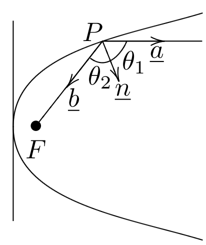
coordinates, then $|PF| = r$, $|P\mathcal{D}| = r\cos\theta + k/e$.
So $|PF| = e|P\mathcal{D}|$ if $r = er\cos\theta + k$. That is, $r(1 - e\cos\theta) = k$, giving the equation

$$r = k/(1 - e\cos\theta),$$

which is the standard form of a conic in polar coordinates.
If $e \geq 1$, the denominator can be zero.

Exercise: What happens if $1 - e\cos\theta < 0$?

5.4. The parabolic mirror



For a 2-dimensional parabolic mirror, rays parallel to the x-axis are reflected through the focus.

Proposition. *Let C be a parabola with focus F, and P a point on C. Let $\underline{n}$ be a unit normal to C at P, and let $\underline{a}$ be a unit vector parallel to the axis of C and let $\underline{b}$ be $\overrightarrow{PF}/|PF|$. Then the angle θ_1 between $\underline{a}$ and $\underline{n}$ equals the angle θ_2 between $\underline{b}$ and $\underline{n}$.*

Proof. Use parametric form, $(x, y) = (a\lambda^2, 2a\lambda)$, and F is $(a, 0)$.
Differentiate: tangent vector is $(2a\lambda, 2a)$.
Unit tangent is given by $\underline{t} = \frac{1}{\sqrt{1+\lambda^2}}(\lambda, 1)$.
Rotate 90 degrees: $\underline{n} = \frac{1}{\sqrt{1+\lambda^2}}(1, -\lambda)$.

$$\begin{aligned}\underline{a} &= (1, 0),\\ \overrightarrow{PF} &= (a(1-\lambda^2), -2a\lambda),\\ |PF|^2 &= a^2(1+\lambda^2)^2.\end{aligned}$$

This implies that

$$\underline{b} = \frac{\overrightarrow{PF}}{|PF|} = \left(\frac{1-\lambda^2}{1+\lambda^2}, \frac{-2\lambda}{1+\lambda^2}\right).$$

Then $\cos\theta_1 = \underline{a} \cdot \underline{n} = \frac{1}{\sqrt{1+\lambda^2}}$.

$$\cos\theta_2 = \underline{b} \cdot \underline{n} = \frac{1}{\sqrt{1+\lambda^2}}\left(\frac{1-\lambda^2}{1+\lambda^2} + \frac{2\lambda^2}{1+\lambda^2}\right) = \frac{1}{\sqrt{1+\lambda^2}}.$$

So $\cos\theta_1 = \cos\theta_2$ and $\theta_1 = \theta_2$. **Q.E.D.**

Mods Geometry handout, week 4 MT07

Standard forms for equations of conics

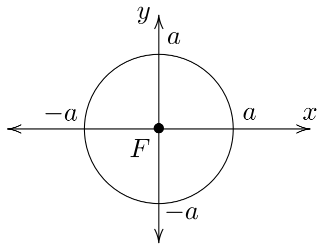
Here are standard forms for the equations of conics in Cartesian coordinates. Every conic can be written in this way after a change of coordinates (rotation and translation).

(a) eccentricity $e = 0$.

The *circle* is $x^2 + y^2 = a^2$.

It has one focus at $(0, 0)$.

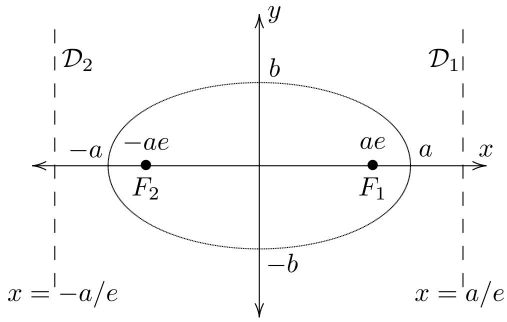
The directrix does not exist.



The standard parametrization is $(x, y) = (a \cos t, a \sin t)$ for $t \in [0, 2\pi)$.

(b) eccentricity $0 < e < 1$.

The *ellipse* is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$,
where $e = \sqrt{1 - b^2/a^2}$,
so that $b^2 = a^2(1 - e^2)$.
There are two foci at $(\pm ae, 0)$,
and two directrices, the lines $x = \pm a/e$.



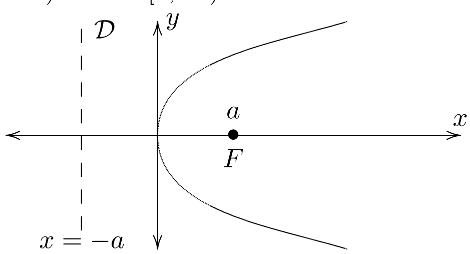
The standard parametrization is $(x, y) = (a \cos t, b \sin t)$ for $t \in [0, 2\pi)$.

(c) eccentricity $e = 1$.

The *parabola* is $y^2 = 4ax$.

It has one focus at $(a, 0)$,

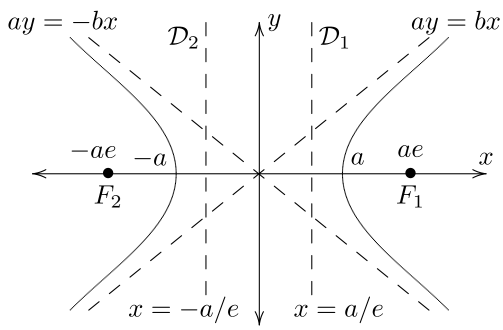
and one directrix $x = -a$.



The standard parametrization is $(x, y) = (at^2, 2at)$ for $t \in \mathbb{R}$.

(d) eccentricity $e > 1$.

The *hyperbola* is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$,
where $e = \sqrt{1 + b^2/a^2}$,
so that $b^2 = a^2(e^2 - 1)$.
There are two foci at $(\pm ae, 0)$,
and two directrices, the lines $x = \pm a/e$.



The hyperbola has two components, parametrized by $(x, y) = (a \cosh t, b \sinh t)$ and $(x, y) = (-a \cosh t, -b \sinh t)$, for $t \in \mathbb{R}$. The lines $ay = \pm bx$ are called the *asymptotes* of the hyperbola. When $e = \sqrt{2}$ and $a = b$ the hyperbola is called *rectangular*.

DDJ